Math 502: Combinatorics II

Midterm Exam, March 13

INSTRUCTIONS: This is an in-class exam; you have 50 minutes to work on the problems. You may not use calculators, cell phones, notes, references, or other aid besides a pen or pencil. Scratch paper will be provided. If you run out of room to write up your answer below a problem statement, continue on the back of that page.

Print your name and sign this exam on the lines below.

Have fun!

SCORING: Each problem is worth the total possible number of points indicated below, where number 6 is a bonus problem. Partial credit will be given for significant progress towards a solution on proof-based problems.

Problem	Score	Points	Problem	Score	Points
1.		8	4.		5
2.		5	5.		2
3.		5	6.		+1

Total:

HONOR PLEDGE: *This exam is my own work. I have not given, received, or used any unauthorized assistance.*

____

Print Name

____

Signature

1. **Symmetric function calculations:** For the following problems, only an answer is required, no explanation or proof.

   (a) (1 point) Write the elementary symmetric function $e_{(2,1)}$ in terms of monomial symmetric functions.

   (b) (1 point) Write the Schur function $s_{(2,1)}$ in terms of monomial symmetric functions.

   (c) (1 point) Write the homogeneous symmetric function $h_{(2,1)}$ in terms of Schur functions.

   (d) (1 point) Write the elementary symmetric function $e_{(2,1)}$ in terms of Schur functions.

(e) (1 point) Write the power sum symmetric function p_3 in terms of elementary symmetric functions.

(f) (1 point) Compute $\langle p_3, m_{(2,1)} \rangle$ where $\langle , \rangle$ is the Hall inner product.

(g) (1 point) Compute $\langle s_{(4,2,1,1)/(2,1)}, s_{(3,1,1)} \rangle$.

(h) (1 point) Compute $\langle s_{(4,2,1,1)}, s_{(2,1)} s_{(3,1,1)} \rangle$.

2. **Jacobi-Trudi bijection:** Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\mu = (\mu_1, \ldots, \mu_j)$ be partitions with μ fitting inside λ, i.e., $\mu_i \leq \lambda_i$ for all i.

   (a) (2 points) Describe a bijection between the semistandard Young tableaux of skew shape λ/μ whose entries are in $\{1, 2, \ldots, m\}$ and the nonintersecting lattice paths from $A_1, \ldots, A_k$ to $B_1, \ldots, B_k$ where $A_i = (\mu_i - i, 1)$ and $B_i = (\lambda_i - i, m)$ for each i (where we set $\mu_i = 0$ if $i > j$). You do not have to prove that your map is a bijection.

   (b) (1 point) Apply your bijection to the following skew tableau of shape $(3, 3, 1)/(1)$:

4		
1	3	4
	2	2

3. **Designs, short answer:** The points and 3-flats of $\mathbb{P}^5_{\mathbb{F}_4} = \mathrm{PG}(5,4)$ form a $2-(v,k,\lambda)$ design.

   (a) (1 point) Find v.

   (b) (1 point) Find k.

   (c) (1 point) Find λ.

   (d) (1 point) Find b, the number of blocks of this design.

   (e) (1 point) Find r, the number of blocks containing a given point.

4. **Difference sets:** Let G be an abelian group with size $v = |G|$, and let $D \subset G$ be a subset of the elements of size $k = |D|$ (not necessarily a subgroup). Then D is a (v, k, λ)-*difference set* if the multiset of elements

$$\{d_i - d_j : d_i, d_j \in D\}$$

contains every non-identity element of G exactly λ times. For instance, $\{0, 1, 3\}$ is a $(7, 3, 1)$-difference set for $G = \mathbb{Z}/7\mathbb{Z}$.

   (a) (1 point) Show that $\{0, 1, 3\}$ is a $(7, 3, 1)$-difference set for $G = \mathbb{Z}/7\mathbb{Z}$.

   (b) (3 points) Suppose D is a (v, k, λ)-difference set for G. Show that the pair $(G, \mathcal{B})$ where $\mathcal{B} = \{D + g : g \in G\}$ is a $2 - (v, k, \lambda)$ design. This design is called the *development* of D.

   (c) (1 point) Compute the development of $\{0, 1, 3\}$ in $\mathbb{Z}/7\mathbb{Z}$.

5. **MOLS:** (2 points) Construct two mutually orthogonal Latin squares of order 5.

6. (**Bonus**, +1 point) Construct four mutually orthogonal Latin squares of order 5.