Math 502: Combinatorics II
Homework 4 - Due March 13

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 10. The maximum possible score on this homework is 10 points. See the syllabus for details.

Problems

1. (2) [3 points] Show that, given four points P_1, P_2, P_3, P_4 with no three collinear in $\mathbb{P}^2_{\mathbb{F}_q}$, and four other points Q_1, Q_2, Q_3, Q_3 with no three collinear, there is a unique automorphism of $\mathbb{P}^2_{\mathbb{F}_q}$ sending P_i to Q_i for all i. (Hint: First prove it in the special case that one of the sets of four points is $(0:0:1), (0:1:0), (1:0:0), (1:1:1)$.)
2. (2) [3 points] Show directly that Pappus's theorem holds in $\mathbb{P}^2_{\mathbb{F}_q}$ for all prime powers q. (You may use the previous problem to simplify your work.)
3. (2+) [4 points] Find all possible parameters of non-trivial designs with 6 points.
4. Let $\mathrm{Gr}(k,n;q)$ be the *finite Grassmannian*, the set of all k-dimensional subspaces of $\mathbb{F}_q^n$. Let $j > k$ and define a j-block to be the set of all k-dimensional subspaces contained in a given j-dimensional subspace.
   (a) (1+) [2 points] Show that the set of points and j-blocks of the Grassmannian forms a 1-design and determine its parameters (v, k, λ).
   (b) (2) [3 points] Show that it does not form a 2-design unless either $k = 1$ or $k = n - 1$.
5. (2) [3 points] A *character* χ of an abelian group A is a homomorphism from A to the multiplicative group of non-zero complex numbers. The *character table* of A is the matrix whose (i, j) entry is the value of the ith character on the jth element of A (for some ordering of the group elements and the characters). Show that, if A is a direct product of cyclic groups of order 2, then its character table is a Hadamard matrix of Sylvester type.
6. (2-) [3 points] Recall that we showed on the last homework that, if a $t-(v, k, \lambda)$ design has an extension, then $k + 1$ divides $b(v + 1)$. Use this to show that if a projective plane of order n has an extension, then n is either 2 or 4. (You may also use the known result that there is no projective plane of order 10, which was verified by computer search.)
7. (1+) [2 points] Construct a set of three mutually orthogonal Latin squares of order 4. (Hint: if you are using the affine plane approach, remember that $\mathbb{F}_4$ is not equal to $\mathbb{Z}/4\mathbb{Z}$!)
8. (2+) [4 points] Construct a pair of orthogonal Latin squares of order 12.