Math 502: Combinatorics II
Homework 3 - Due Feb 28

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 10. The maximum possible score on this homework is 10 points. See the syllabus for details.

Problems

1. (2-) [3 points] Expand the product of Schur functions $s_{(2,2)} \cdot s_{(2,1)}$ in the Schur basis without using a computer. (Hint: the Yamanouchi condition for testing whether a tableau is Littlewood-Richardson may come in handy.)

2. (2-) [3 points] Adapt the argument shown in class to prove the skew version of the Jacobi-Trudi identity, namely,
$$s_{\lambda/\mu} = \det\left(h_{\lambda_i - \mu_j + j - i}\right)_{i,j=1}^{n}$$
where n is the number of parts of λ.

3. (1+) [2 points] Construct a $1-(10,6,3)$ design.

4. (2) [3 points] (Cameron and Van Lint problem 1) Let X be the set of 1-dimensional subspaces of $\mathbb{F}_q^{n+1}$, and let $\mathcal{B}$ be the set of $d+1$-dimensional subspaces of $\mathbb{F}_q^{n+1}$. (Note that these are the points and d-planes in $\mathbb{P}^n_{\mathbb{F}_q}$ respectively.) Prove that $(X, \mathcal{B})$ is a 2-design and determine its parameters v, k, λ.

5. (3-) [8 points] (Cameron and Van Lint problem 5)

6. (3-) [8 points] (Cameron and Van Lint problem 8)

7. (2+) [4 points] (Cameron and Van Lint problem 11)

8. (2) [3 points] Prove that there is a unique $2-(7,3,1)$ design up to isomorphism.

9. (1+) [2 points] Prove that a necessary condition for a $t-(v,k,\lambda)$ design with b blocks to have an extension is that $(v+1)$ divides $b(k+1)$.

10. Let G be an abelian group with size $v = |G|$, and let $D \subset G$ be a subset of the elements of size $k = |D|$ (not necessarily a subgroup). Then D is a (v,k,λ)-*difference set* if the multiset of elements
$$\{d_i - d_j : d_i, d_j \in D\}$$
contains every non-identity element of G exactly λ times. For instance, $\{0,1,3\}$ is a $(7,3,1)$-difference set for $G = \mathbb{Z}/7\mathbb{Z}$.

    (a) (2) [3 points] Suppose D is a (v,k,λ)-difference set for G. Show that the pair $(G, \mathcal{B})$ where $\mathcal{B} = \{D + g : g \in G\}$ is a $2-(v,k,\lambda)$ design. This design is called the *development* of D.

    (b) (1) [1 point] Compute the development of $\{0,1,3\}$ in $\mathbb{Z}/7\mathbb{Z}$.

11. (From the 2015 Prove it! Math Academy Admissions Test.) At Mahalo Yogurt[1], there are seven flavors of soft-serve frozen yogurt. You are allowed to choose any number of flavors to put in your yogurt cup. Equal amounts of each flavor are put into each cup, and the order and orientation of which flavor is placed where when is irrelevant.

    (a) (1-) [1 point] You plan on going to Mahalo's once every day for a week. You would like to figure out which combination of two flavors tastes the best together. However, you realize that there are too many combinations to try them all out in this period of time. How many different combinations of two distinct flavors are there?

[1] Mahalo Yogurt was an actual frozen yogurt shop in Fort Collins back in 2015, on the corner of Elizabeth and Shields, but it has since closed.

(b) (1-) [1 point] Since you can't try every possible pairing of flavors in seven tries by just ordering two scoops at a time, you decide you'll buy three flavors per day! You want to pursue a method that will maximize the different flavor combinations you can experience over the course of the week. In particular you demand the following:

   i. Throughout the week, you try each individual flavor an equal number of times.
   ii. Every pairing of flavors is tried once. For example, somewhere throughout the week you must try chocolate and vanilla together in the same cup (along with some third flavor).
   iii. No pairing of flavors is tried more than once. For example, if you try chocolate and vanilla together on Monday, then throughout the rest of the week you might eat chocolate and you might eat vanilla, but you will never again eat them together in the same cup, since you already tried that pairing.

   Show that it is possible to devise a tasting plan that fits the above requirements.

(c) (1+) [2 points] A particularly hungry and indecisive friend would like to try each pair of flavors **more** than once. Devise a plan where you order **four** flavors per day that allows you to try each pairing of flavors exactly twice.

(d) (1+) [2 points] Your especially particularly hungry and indecisive friend is overjoyed regarding the last two plans you showed him. In excitement, he asks you to devise a plan where he can order **five** flavors per day so that he can try each pairing of flavors exactly three times. Explain to him why, sadly, this is not possible!