Math 502: Combinatorics II
Homework 2 - Due Feb 14

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 10. The maximum possible score on this homework is 10 points. See the syllabus for details.

# Problems

1. (0) [0 points, MANDATORY - if you do not hand in anything for this question you get a 0 for this homework.] Describe, in a few sentences, your plan for the final project. What topic are you going to write about? How do you plan to start your investigation?

2. (1+) [2 points] Prove that any antisymmetric function $f(x_1, \ldots, x_n)$ is divisible by $x_1 - x_2$.

3. (1+) [2 points] Apply the RSK algorithm to the two-line array

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 3 & 3 \\ 1 & 2 & 2 & 1 & 3 & 1 & 2 \end{pmatrix}$$

   to obtain a pair of semistandard Young tableaux of the same shape.

4. (1+) [2 points] Compute $\langle s_{3,2}, h_{2,2,1} \rangle$ (without using a computer). Explain your reasoning.

5. (1+) [2 points] Prove that $h_\mu = \sum_\lambda K_{\lambda\mu} s_\lambda$ for any partition $\mu$, where $K_{\lambda\mu}$ are the Kostka numbers. (Hint: use the Hall inner product starting with the identity $s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu$.)

6. (2) [3 points] Write the power sum symmetric function $p_4$ in terms of the $e$ basis, the $h$ basis, and the $s$ basis.

7. (2) [3 points] Write the skew Schur function $s_{(3,2)/(2)}$ in terms of the (straight shape) Schur basis $\{s_\lambda\}$. Also write the product $s_{(1)}s_{(2)}$ in terms of the Schur basis. What do you notice?

8. (2+) [4 points] (Stanley problem 7.3) Expand the power series $\prod_{i \geq 1}(1 + x_i + x_i^2)$ in terms of the elementary symmetric functions.

9. (2+) [4 points] (Stanley problem 7.11) Let $q$ be an indeterminate. Find the Schur function expansion of $\sum_{\mu \vdash n} q^{\ell(\mu)-1} m_\mu$.

10. (4-) [10 points] For a partition $\lambda$ of $n$, write $Y(\lambda)$ for the set of squares in its Young diagram. For a square $s \in Y(\lambda)$, define its **hook length** $h(s)$ to be the number of squares that are either equal to $s$, to the right of $s$ in its row, or above $s$ in its column (in French notation).

    Prove the **hook length formula**: that the number of standard Young tableaux (that is, semistandard Young tableaux having content $(1, 1, 1, \ldots)$) of shape $\lambda$ is equal to

$$\frac{n!}{\prod_{s \in Y(\lambda)} h(s)}$$

    (Note: an acceptable answer to problem 1 is "I will read about and write up a proof or two of the Hook Length Formula.")