Math 502: Combinatorics II
Homework 1 - Due Jan 31

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 10. The maximum possible score on this homework is 10 points. See the syllabus for details.

Problems

1. (1+) [2 points] Define $H(t) = \sum_{n=0}^{\infty} h_n t^n$ to be the generating function of the homogeneous symmetric functions h_n in the set of variables $X = \{x_1, x_2, \ldots\}$. Prove the identity
$$H(t) = \prod_{n=1}^{\infty} \frac{1}{1 - x_n t}.$$

2. (1+) [2 points] Define $E(t) = \sum_{n=0}^{\infty} e_n t^n$ to be the generating function of the homogeneous symmetric functions e_n in the set of variables $X = \{x_1, x_2, \ldots\}$. Prove the identity
$$E(t) = \prod_{n=1}^{\infty} (1 + x_n t).$$

3. (1+) [2 points] Draw the Hasse diagram of the poset $(\mathrm{Par}(7), \preceq)$ where $\mathrm{Par}(7)$ is the set of all partitions of 7 and $\preceq$ is dominance order. (Hint: it is not a graded poset!)

4. (1+) [2 points] Write the symmetric polynomial $x_1^4 + x_2^4 + x_3^4 - x_1x_2 - x_1x_3 - x_2x_3$ in $\Lambda_{\mathbb{Q}}(X_3)$ in terms of the m basis, the e basis, and the p basis.

5. (1+) [2 points] We will see later that the product of two Schur functions is Schur-positive, that is, if we write $s_\lambda \cdot s_\mu$ in terms of the Schur basis as $\sum c_{\lambda\mu}^{\nu} s_\nu$, then the coefficients $c_{\lambda\mu}^{\nu}$ are positive integers. In the same sense, is the product of two monomial symmetric functions always monomial-positive? Is the product of two elementary symmetric functions always elementary-positive? Why or why not, in each case?

6. (1+) [2 points] Show that $h_\lambda = \sum N_{\lambda\mu} m_\mu$ where $N_{\lambda\mu}$ is the number of matrices with nonnegative integer entries whose row sums are $\lambda_1, \lambda_2, \ldots$ and whose column sums are $\mu_1, \mu_2, \ldots$.

7. (2) [3 points] Let $(\mathrm{Par}(n), \preceq)$ be the poset of partitions of n under dominance order. Prove that this poset is a lattice.

8. (2+) [4 points] Prove that $\mathrm{Par}(n, \preceq)$ is a self-dual poset.

9. (5) [∞ points] The *chromatic symmetric function* of a (undirected) graph G on vertex set $V = \{v_1, \ldots, v_n\}$ can be defined as follows. A *proper coloring* is a map $c : V \to \mathbb{N}_{\geq 0}$ such that no two adjacent vertices in the graph are assigned the same label by c. The *monomial* associated to c is
$$x_c := x_{c(v_1)} x_{c(v_2)} \cdots x_{c(v_n)}.$$

   Finally, the chromatic symmetric function X_G is defined to be $\sum_c x_c$ where the sum is over all proper colorings c of G.

   Many of these chromatic symmetric functions have nice positivity properties in terms of common symmetric function bases. One such class of them is given by "incomparability graphs". Given a poset P, the incomparability graph is the graph on the elements of the poset in which an edge is drawn between x and y if they are incomparable, that is, $x \not\leq y$ and $y \not\leq x$.

   A poset is said to be $(3+1)$-free if it contains no induced subposet isomorphic to the direct sum of a 1-chain and a 3-chain. Show that if G is the incomparability graph of a $(3+1)$-free poset, then X_G is e-positive, that is, it can be written as a sum of elementary symmetric functions with positive integer coefficients.

   This is known as the *Stanley-Stembridge conjecture*.