Math 502: Combinatorics II
Final Exam Review Worksheet

More problems will be added to this document for each day of review!

1 Symmetric functions

1. The manager of Burger King wants to know how many ways he place 8 identical burgers in 8 of the squares of a 4 by 4 grid in such a way that every row and column contains exactly 2 burgers. Explain how you can come up with the answer quickly using the symmetric functions package in Sage. Then, obtain the answer using Sage!
2. How many ways can the manager of Burger King arrange his burgers if he is allowed to stack more than one burger in a given square of the grid? He still wishes to have every row and column contain exactly 2 burgers, but now a given square can contain more than one burger.
3. State and prove the Fundamental Theorem of Symmetric Function Theory.
4. Express the power sum symmetric function $p_{(3,1)}$ in terms of the monomial basis.
5. Express $p_{(3,1)}$ in terms of the elementary basis.
6. Express $p_{(3,1)}$ in terms of the homogeneous basis.
7. Express $p_{(3,1)}$ in terms of the Schur basis.
8. Express the homogeneous symmetric function $h_{(5)}$ in terms of the Schur basis.
9. Express the elementary symmetric function $e_{(5)}$ in terms of the Schur basis.
10. Apply the involution ω to the monomial symmetric function $m_{(2,1)} + m_{(3)}$, and express your answer in the monomial basis.
11. Express the product $s_{(3,3,2)} \cdot h_{(2)}$ in terms of the Schur basis.
12. Compute $\langle h_{(1,1,1)}, h_{(2,1)} \rangle$.
13. Compute $\langle s_{(5,3,2,1)/(3,2,1)}, s_{(3,2)} \rangle$.
14. Express the skew Schur function $s_{(3,3,1)/(2,2)}$ in the monomial basis.
15. State and prove the Jacobi-Trudi identity.

2 Designs

1. Prove that, in any $t-(v,k,\lambda)$ design, the number of blocks is $b = \lambda\binom{v}{t}/\binom{k}{t}$.
2. Prove that, in any $t-(v,k,\lambda)$ design, we have $bk = vr$.
3. Prove that any t-design is also an s-design for any $s \leq t$.
4. A cook at the local Burger King is bored; there is quite a lack of customers due to the novel coronavirus, and only the drive-through window is open anyway. He decides to test which pairs of toppings for a burger taste best together. There are 11 different toppings available to choose from: cheese, ketchup, mustard, pickles, lettuce, relish, onions, tomatoes, bacon, mayonnaise, and jalapenos.

   To make his experiment more efficient, he decides to put 5 toppings on each burger that he taste-tests. Enough to taste multiple combinations, but not so much as to be overwhelmed by different tastes. He also wants to make sure he tests every pair exactly twice, to get a sense of how that combination tastes under different circumstances.

Can he design a valid experiment? If so, how many burgers would he need to eat to complete his experiment?

5. State and prove Fisher's Inequality for 2-designs.
6. Define an abstract projective plane in terms of design parameters.
7. Show that any projective plane is self-dual.
8. The points and 2-flats of the projective geometry $\mathbb{P}^4_{\mathbb{F}_3}$ form a $2-(v,k,\lambda)$ design. Find v,k,λ,b,r.
9. Construct four mutually orthogonal Latin squares of size 5 using finite projective geometry.
10. Construct an extension of the Fano plane. As a 3-design, what are its parameters?

3 Strongly Regular Graphs

1. Show that the Petersen graph is strongly regular, and find its parameters (n,k,λ,μ).
2. Define a rank three graph, and show that $L_2(3)$ is rank three (and hence strongly regular).
3. Let G be a graph with adjacency matrix A. Show that it is strongly regular if and only if A^2 can be written as a linear combination of A, I, and $J-I-A$ (where J is the all-ones matrix and I is the identity matrix).
4. If G is strongly regular with parameters (n,k,λ,μ), what are the coefficients of A, I, and $J-I-A$ in the expansion of A^2 described above? Express your answers in terms of n,k,λ,μ.
5. Prove that, if G is strongly regular with parameters (n,k,λ,μ), then
$$k(k-\lambda-1)=(n-k-1)\mu.$$
Prove this in two ways: first combinatorially, and second with linear algebra (see the problems above).
6. Show that A is the adjacency matrix of a strongly regular graph if and only if has exactly two other eigenvalues besides k. (Hint: For the forward direction, consider the problems above. For the reverse direction, show that if its other two eigenvalues are r and s, then $(A-rI)(A-sI)$ is a multiple of J, and then conclude the result using problem 3 in this list.)

4 Codes

1. Prove that, in a linear code, the minimum distance equals the minimum weight. What is the error-correcting capacity of a linear code with minimum weight 5?
2. Write down both the parity check matrix and a generator matrix for the Hamming code of dimension 4 in $\mathbb{F}_2^7$.
3. Prove that every Hamming code is a perfect 1-error-correcting code.
4. Give an example of a self-dual linear code that is at least 1-error-correcting.
5. Consider the cyclic code in $\mathbb{F}_2^6$ generated by the polynomial x^2+x+1. Write down its generator matrix and parity check matrix.