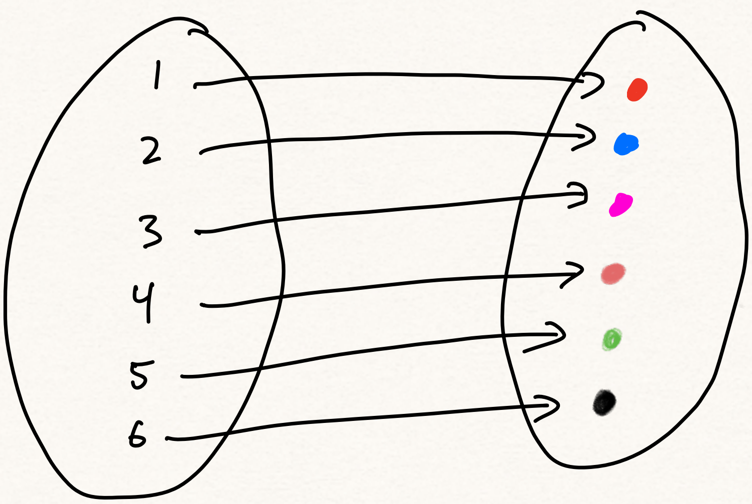
Counting basics: What does it mean to count something?

E.g. How many dots are there?

A way of counting is an assignment of the numbers $1, 2, \ldots, n$ to each object such that each object is counted exactly once. This is a bijection from $\{1, \ldots, n\}$ to the set:



This is just one of many ways we can count these six dots one by one!

Q: How many different ways can we count the 6 dots? (i.e. how many bijections) (Hint: start w/ 2 dots)

Recursion: a way of counting by starting w/ smaller similar sets and building on that answer

$R_n = \#$ ways to count an n-elt set.

- n ways to choose which is counted first
- R_{n-1} ways to count the rest

$\Rightarrow R_n = n \cdot R_{n-1}$ (Multiplication and addition principle: "and" = $\times$, "or" = $+$)

$R_1 = 1$

$\Rightarrow R_n = n!$

Cardinality and bijections

Def: A, B sets. $f: A \to B$ is **bijective** if it is:

(1) **Injective** (one-to-one): $\forall x, y \in A$, $f(x) = f(y) \Rightarrow x = y$

(2) **Surjective** (onto): $\forall y \in B$, $\exists x \in A$, $f(x) = y$.

THM: f is bijective iff it has an inverse $g: B \to A$;

$$\left.\begin{aligned} g \circ f &= id_A \\ f \circ g &= id_B \end{aligned}\right\} \text{write } g = f^{-1}$$

<u>Def:</u> Write $A \cong B$, or $|A| = |B|$, if there exists a bijection $f: A \to B$.

<u>Prop:</u> $\cong$ is:

- Reflexive: $A \cong A$
- Symmetric: $A \cong B \Leftrightarrow B \cong A$
- Transitive: $A \cong B$ and $B \cong C \Rightarrow A \cong C$

<u>Pf:</u>
- id_A
- $f: A \to B$ bijection $\Rightarrow f^{-1}: B \to A$ bijection.
- $A \xrightarrow{f} B \xrightarrow{g} C$ bijections $\Rightarrow g \circ f$ bijection (inverse: $f^{-1} \circ g^{-1}$)

$\square$

Thus $\cong$ is an <u>equivalence relation</u> and sets can be sorted into <u>equivalence classes</u> based on size. These equiv classes are called <u>cardinalities</u>.

Cardinality of A: $|A|$ is the collection of all sets in bijection with A.

<u>Ex:</u> $\underline{2} = [\{1,2\}, \{\bullet, \bullet\}, \ldots] = |\{\circ, \bullet\}|$

<u>Infinite cardinalities:</u> $|\mathbb{N}| \neq |\mathbb{R}|$

$$|\mathbb{N}| = |\mathbb{Q}|$$

Addition of cardinalities:

$$|A| + |B| = |A \sqcup B|$$

disjoint union: $(A \times \{0\}) \cup (B \times \{1\})$

Ex: $\underline{2} + \underline{3} = |\{\circ, \bullet\} \cup \{a, b, c\}| = |\{\circ, \bullet, a, b, c\}| = \underline{5}$

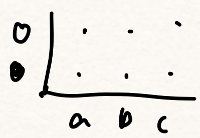
Pf that this is well-defined: Need to show that if $C \cong A$ and $D \cong B$ then

$$C \sqcup D \cong A \cup B.$$

Mult. of cardinalities:

$$|A| \cdot |B| = |A \times B|$$

Ex: $\underline{2} \cdot \underline{3} = |\{\bullet, \circ\} \times \{a, b, c\}| = \underline{6}$



"choose one thing from A <u>and</u> <u>then</u> one thing from B"