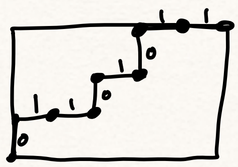
# Lectures 4-5

Other interpretations of $\binom{n}{k}$:

- # 0,1 - sequences with $k$ zeroes of length $n$
- # paths from $(0,0)$ to $(k, n-k)$ using steps of $+(1,0)$ or $+(0,1)$




$\longleftrightarrow$ 01101011

- Coeff of $x^k y^{n-k}$ in $(x+y)^n$ (Binomial Theorem)

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<u>Proof</u> that $\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$:

LHS counts ways of choosing a handful of $k$ M&M's from a bag of M&M's that come in $n$ colors (e.g. $\left(\!\binom{6}{10}\!\right)$).

RHS counts sequences of $k$ 0's and $n-1$ 1's.

<u>Bijection</u>: Choose an ordering of the colors, e.g.

| Red | Orange | Yellow | Green | Blue | Brown |
|---|---|---|---|---|---|
| OO | | O | OOO | OOOO | |

and for a given handful, sort them by color and then put dividers (1's) between the colors:

00|   | 0 | 000|0000|

$\downarrow$

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This gives a sequence of $k$ 0's and $n-1$ 1's. To show this process is a bijection, we note that every step is reversible; given a sequence, think of the 1's as dividers between the colors and the 0's as the M&M's of each color.

QED

Summary chart of combinations:

| choose $k$ things from $n$ | Repeats allowed | Repeats not allowed |
|---|---|---|
| order matters | $n^k$ | $(n)_k$ |
| order doesn't matter | $\left(\!\binom{n}{k}\!\right)$ | $\binom{n}{k}$ |

Can extend this chart to Rota's "Twelvefold Way."

Twelvefold Way: Count maps $f:[k]\to[n]$

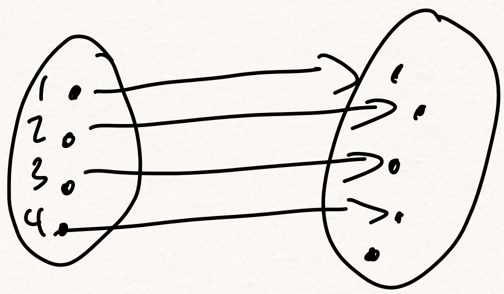
| | Any | Injective | Surjective |
|---|---|---|---|
| $[k]$ dist, $[n]$ dist | $n^k$ | $(n)_k$ | A |
| $[k]$ indist, $[n]$ dist. | $\left(\!\binom{n}{k}\!\right)$ | $\binom{n}{k}$ | B |
| $[k]$ dist, $[n]$ indist. | C | D | E |
| $[k]$ indist. $[n]$ indist. | F | G | H |

dist: "distinguishable" - ordinary set of distinct elts

indist: "indistinguishable" - we only consider maps different up to relabeling the indistinguishable elts

Want to fill in A – H.

Easiest: D and G. Want injective maps to indistinguishable elts:




only matters that $k \le n$

So $D = G = \begin{cases} 0 & k > n \\ 1 & k \le n. \end{cases}$

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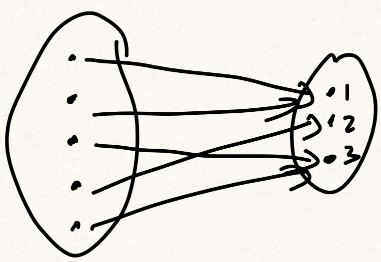
**B**: Map from $[k]$ indist $\to [n]$ dist, <u>surjective</u>:




Same as choosing a multiset from $[n]$ of size $k$ but every elt of $n$ has to be used at least once.

So we pick 1 of each and then choose $k-n$ more...

$$\Rightarrow \quad B = \left(\!\!\binom{n}{k-n}\!\!\right)$$

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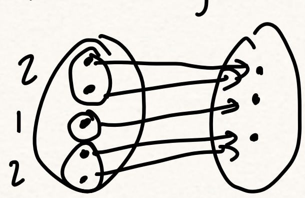
**H** Both indistinguishable, surjective:




Categorized by sizes of preimages; <u>partitions</u> into $n$ <u>parts</u>

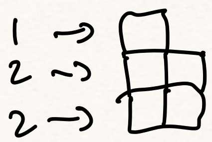
**Def:** A <u>partition</u> of a positive integer $k$ is a way of writing $k$ as a sum of positive integers (no particular order, so we canonically choose decreasing:

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n), \quad \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n,$$

$$\sum \lambda_i = k.$$

Write $\ell(\lambda) = n$ and each $\lambda_i$ is a <u>part</u>

**Notation:** $(2,2,1)$ partition of 5




← <u>Young diagram</u>

**Def:** $p(k)$ = # partitions of $k$

$p(k,n)$ = # partitions of $k$ into $n$ parts.

**Ex:** Compute $p(5)$ and $p(5,2)$.

$\# = p(k,n)$.

**Recursion:** $p(k,n) = p(k-1, n-1) + p(k-n, n)$

(Proof in class)

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F If map no longer has to be surjective, it's a partition of $k$ into at most $n$ parts.

$$p(k,0) + p(k,1) + \dots + p(k,n).$$

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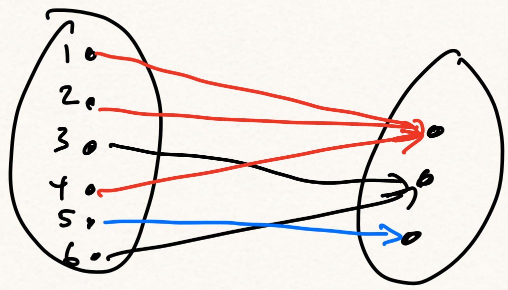
E Surj. map




corresponds to the set partition

$$\{\{1,2,4\}, \{3,6\}, \{5\}\}$$ based on the preimages of the points.

Def: A set partition of $A$ is a collection of disjoint nonempty subsets (blocks) $B_1, \dots, B_r$ whose union is $A$.

Notation: $\{124, 36, 5\}$ – order L to R by min elt, order each block increasing.

Def: Stirling number (of the second kind): $S(k,n) = \#$ set partitions of $[k]$ into $n$ blocks.

$\boxed{E = S(k,n)}$.

Recursion (prove in class):

- $S(k,1) = 1$
- $S(k,k) = 1$
- $S(k,n) = S(k-1,n-1) + n \cdot S(k-1,n)$

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C By similar logic to F, we have

$$C = S(k,1) + S(k,2) + \dots + S(k,n)$$

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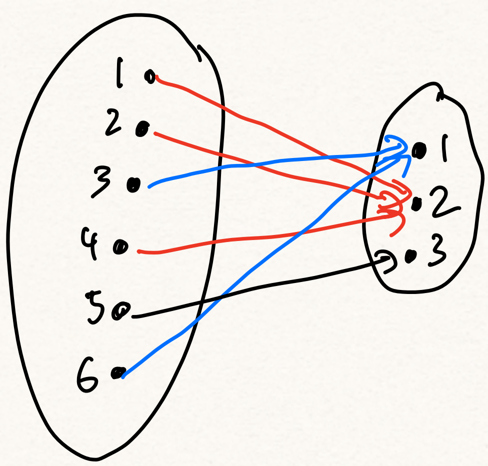
A




$\to (36 \mid 124 \mid 5)$

ordered set partition — a set partition w/

an ordering on the blocks.

$n!$ ways to order the blocks

$$\Rightarrow \boxed{A = n!\, S(k,n)}.$$