s to h: Jacobi-Trudi formula

Thm: $s_\lambda = \det\left(h_{\lambda_i - i + j}\right)_{i,j}$ (i,j btwn 1 and $\ell(\lambda)$).

Ex: $s_{(2,1)} = \det\begin{pmatrix} h_2 & h_3 \\ h_0 & h_1 \end{pmatrix}$

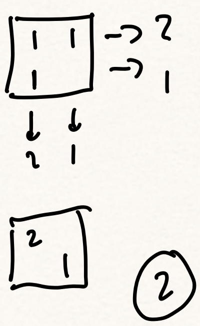
Note: h_2, h_1 go on diags, then row subscripts increase by 1 L to R

$= h_2 h_1 - h_3$

$= (2m_{(2,1)} + m_3 + 3m_{111}) - (m_{111} + m_{21} + m_3)$

$= m_{(2,1)} + 2m_{111}$ ✓

$N_{(2,1),(2,1)}$:



Ex: $s_{(3,1,1)} = \det\begin{pmatrix} h_3 & h_4 & h_5 \\ h_0 & h_1 & h_2 \\ 0 & h_0 & h_1 \end{pmatrix} = h_{311} + h_5 - h_{41} - h_{32}$

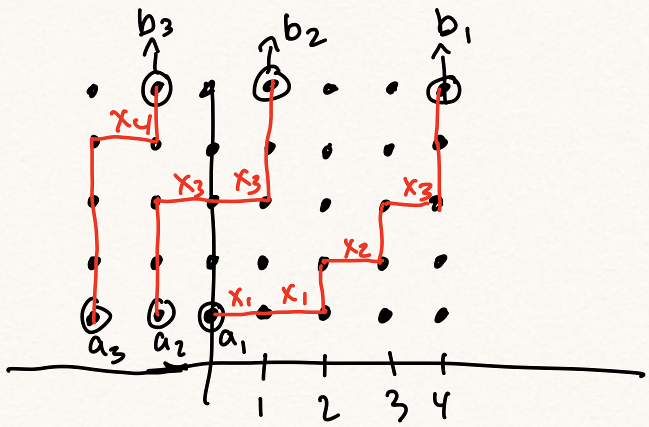
$\boxed{\begin{array}{l} h_{-i} = 0 \\ h_0 = 1 \end{array}}$

Pf: We'll use a weighted version of Lindström-Gessel-Viennot (nonint. lattice paths.)

Sources: $a_1, \ldots, a_\ell$, sinks $b_1, \ldots, b_\ell$

$a_i = (1-i, 1)$ $\qquad$ $b_i = (1-i+\lambda_i, \infty)$

(on lattice grid, ∞ means "sufficiently high height")



Ex: $\lambda = (4,2,1)$

Weighted edges for digraph: weight all up-arrows by 1, all right-arrows at height i by x_i.

Weighted LGV: weight of an n-path is product of the weights of all the edges.

Then $$\sum_{\substack{n\text{-Path} \\ P:\underline{a}\to\underline{b}}} wt(P)\cdot sgn(P) = \sum_{\substack{\text{nonint-} \\ n\text{-paths} \\ P:\underline{a}\to\underline{b}}} wt(P)\cdot \cancel{sgn(P)}$$

all permutations id for nonint here.

$$= \det(w_{ij})$$

where $w_{ij} = \sum$ wts of all paths $a_i \to b_j$.

Here, $w_{ij} = \sum$ monomials of deg $\begin{matrix}(b_j \text{ x coord}) \\ -(a_i \text{ x coord})\end{matrix}$

$$= \sum \text{monomials deg } (1-j+\lambda_j-(1-i))$$

$$= \sum \text{monomials deg } (\lambda_j+i-j). = h_{\lambda_j+i-j}$$

Transpose is JT matrix. ✓

(Transpose has same det)

So, we've shown:

$$\det\left(h_{\lambda_i+j-i}\right) = \sum_{\substack{\text{nonint} \\ n\text{-paths } P \\ \underline{a} \to \underline{b}}} wt(P)$$

We now show there is a weight-preserving bijection

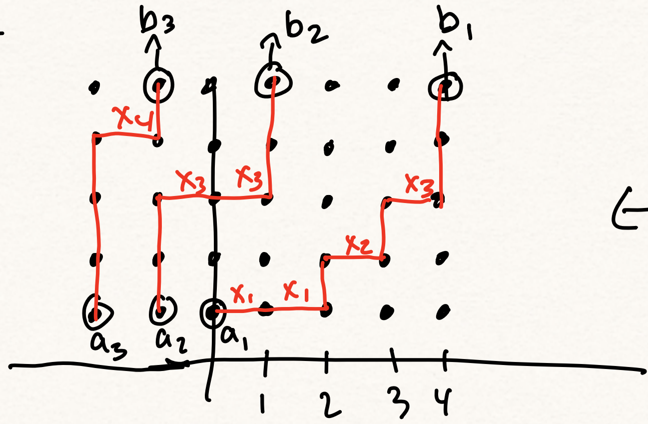
$$\left\{\begin{matrix}\text{nonint } n \text{ paths} \\ \underline{a} \to \underline{b}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{SSYT's shape} \\ \lambda\end{matrix}\right\}$$

$$P \longmapsto T$$

$$wt(P) = \text{monomial } wt(T)$$

Bijection: heights of horiz. edges on $a_i \to b_i$ are i-th row of T.

Ex:



$\longleftrightarrow$

4			
3	3		
1	1	2	3

Pf that this map is <u>well-defined</u>:

- Since heights of each path $a_i \to b_i$ are weakly increasing, rows of T are weakly increasing.
- Since paths are nonintersecting, heights of each successive path are larger than those of the next
  $\Rightarrow$ cols are strictly increasing.

Pf that it is wt-preserving: by def.

Pf that it is a bijection: Reversible. ✓

QED.