Algebraic definition of Schur functions

Antisymmetric polynomial:

$f(x_1, x_2, \ldots, x_n)$ s.t. $\pi f = \mathrm{sgn}(\pi) \cdot f$.

Ex: $f(x_1, x_2) = x_1^2 - x_2^2$

Ex: $f(x_1, x_2, x_3) = (x_1 - x_2)(x_2 - x_3)(x_1 - x_3)$

(Van Der Monde determinant)

Fact: $$\det \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{pmatrix} = \prod_{1 \le i < j \le 4} (x_i - x_j) = \Delta_4$$

Thm: All antisymm. poly's f in n vars are divisible by Δ_n, and in fact $\frac{f}{\Delta_n}$ is symmetric.

Pf: Setting $x_i = x_j$ in f yields 0,

so $x_i - x_j \mid f$. So $\Delta_n \mid f$.

Now, $$\pi \cdot \left(\frac{f}{\Delta_n}\right) = \frac{\pi \cdot f}{\pi \cdot \Delta_n} = \frac{\mathrm{sgn}(\pi) f}{\mathrm{sgn}(\pi) \Delta_n} = \frac{f}{\Delta_n}$$

so $\frac{f}{\Delta_n}$ is symmetric. $\square$

$A_R(x_1,\dots,x_n)$ = module of antisymm. poly's w/ coeffs in R.

<u>NOT</u> a ring or algebra – product of two antisym's is <u>symmetric</u>.

But <u>is</u> a vector space / R-module – closed under $+$.

<u>Basis</u>: Monomial antisymmetrics

$$a_\lambda = x_1^{\lambda_1} x_2^{\lambda_2} + \dots + x_k^{\lambda_k} \pm \text{ all similar terms}$$

sign determined by permutation

defined for <u>strict</u> partitions λ.

<u>Ex</u>: $a_{(5,3,1)}(x_1,x_2,x_3) = x_1^5 x_2^3 x_3 - x_1^5 x_3^3 x_2 - x_2^5 x_1^3 x_3 + x_2^5 x_3^3 x_1 + x_3^5 x_1^3 x_2 - x_3^5 x_2^3 x_1$.

Note: $a_{(2,1,0)} = (x_1 - x_2)(x_2 - x_3)(x_1 - x_3)$

<u>Def ②</u>: $$s_\lambda(x_1,\dots,x_n) = \frac{a_{\lambda+\rho}(x_1,\dots,x_n)}{a_\rho(x_1,\dots,x_n)}$$

where $\rho = (n-1, n-2, \dots, 3, 2, 1, 0)$.

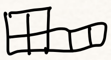
Why equiv. to comb. def?

Lemma: $a_\lambda = \det \begin{pmatrix} x_1^{\lambda_1} & x_1^{\lambda_2} & \cdots & x_1^{\lambda_n} \\ x_2^{\lambda_1} & x_2^{\lambda_2} & \cdots & x_2^{\lambda_n} \\ & \vdots & & \\ x_n^{\lambda_1} & x_n^{\lambda_2} & \cdots & x_n^{\lambda_n} \end{pmatrix}$

Pf: This det is

$$\sum_{\pi \in S_n} (-1)^{\mathrm{inv}(\pi)} x_{\pi(1)}^{\lambda_1} \cdots x_{\pi(n)}^{\lambda_n} = a_\lambda. \qquad \square$$

So the claim is the Schur function is this ratio of determinants. Pf sketch ①: show $a_\rho s_\lambda = a_{\lambda+\rho}$:

Ex: $\lambda =$ , two vars x_1, x_2:

$$\underbrace{\begin{array}{l} 2\,2 \\ 1\,1\,1\,1\,1 \end{array} \quad \begin{array}{l} 2\,2 \\ 1\,1\,1\,1\,2 \end{array} \quad \begin{array}{l} 2\,2 \\ 1\,1\,1\,2\,2 \end{array} \quad \begin{array}{l} 2\,2 \\ 1\,1\,2\,2\,2 \end{array}}_{SSYT(\lambda)}$$

$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

$$\underbrace{x_1^5 x_2^2 \qquad x_1^4 x_2^3 \qquad x_1^3 x_2^4 \qquad x_1^2 x_2^5}_{s_\lambda(x_1, x_2)}$$

$$\det\begin{pmatrix} x_1 & 1 \\ x_2 & 1 \end{pmatrix} \cdot x_1^5 x_2^2 + \det\begin{pmatrix} x_1 & 1 \\ x_2 & 1 \end{pmatrix} \cdot x_1^4 x_2^3 + \det\begin{pmatrix} x_1 & 1 \\ x_2 & 1 \end{pmatrix} \cdot x_1^3 x_2^4 + \det\begin{pmatrix} x_1 & 1 \\ x_2 & 1 \end{pmatrix} \cdot x_1^2 x_2^5 \quad \leftarrow a_\rho$$

$$\|$$

$$\det\begin{pmatrix} x_1^6 & x_1^5 \\ x_2^3 & x_2^2 \end{pmatrix} + \det\begin{pmatrix} x_1^5 & x_1^4 \\ x_2^4 & x_2^3 \end{pmatrix} + \det\begin{pmatrix} x_1^4 & x_1^3 \\ x_2^5 & x_2^4 \end{pmatrix} \overset{\text{want}}{=} \det\begin{pmatrix} x_1^6 & x_1^2 \\ x_2^6 & x_2^2 \end{pmatrix}$$

Full pf of this form in Eric Egge's book on symmetric functions.

Pf sketch 2 (Jacobi): Relate to Jacobi-Trudi determinant formula, $s_\lambda = \det(h_{\lambda_i - i + j})$ (next lecture)

Show $\det(x_i^{j-1}) \cdot \det(h_{\lambda_i - i + j}) = \det(x_i^{\lambda_j})$

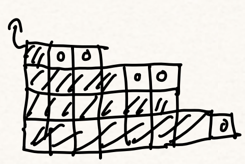
Pf sketch 3 (cleanest but highest level):
(Weyl character formula for GL_n) + (crystal base theory)

Pf sketch 4: See Stanley for theoretical pf

Pf 5 (Proctor 1987)

Lemma: Starting w/ alg. def of s_λ, we have

$$s_\lambda(x_1, \ldots, x_n) = \sum_{\substack{\mu \text{ s.t.} \\ \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \lambda_n}} s_\mu(x_1, \ldots, x_{n-1}) \, x_n^{|\lambda| - |\mu|}$$



"horizontal strip"

Pf by ex: $\lambda = (4,2,1)$:

$$\frac{a_{\lambda+\rho}}{a_\rho} = \frac{\det\begin{pmatrix} x^{4+2} & y^{4+2} & z^{4+2} \\ x^{2+1} & y^{2+1} & z^{2+1} \\ x^{1+0} & y^{1+0} & z^{1+0} \end{pmatrix}}{\det\begin{pmatrix} x^2 & y^2 & z^2 \\ x^1 & y^1 & z^1 \\ x^0 & y^0 & z^0 \end{pmatrix}}$$

set $z=1$, subtract last col from all other cols,

then divide by $x-1$, $y-1$:

$$\det\begin{pmatrix} x^5+x^4+x^3+x^2+x+1 & y^5+y^4+y^3+y^2+y+1 & 1 \\ x^2+x+1 & y^2+y+1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \Big/ \det\begin{pmatrix} x+1 & y+1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Now in each det take consecutive differences of rows:

$$\det\begin{pmatrix} x^5+x^4+x^3 & y^5+y^4+y^3 & 0 \\ x^2+x & y^2+y & 0 \\ 1 & 1 & 1 \end{pmatrix} \Big/ \det\begin{pmatrix} x & y & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \det\begin{pmatrix} x^5+x^4+x^3 & y^5+y^4+y^3 \\ x^2+x & y^2+y \end{pmatrix} \Big/ \det\begin{pmatrix} x & y \\ 1 & 1 \end{pmatrix}$$

$$= \sum_{\substack{4 \geq \mu_1 \geq 2 \\ 2 \geq \mu_2 \geq 1}} \det\begin{pmatrix} x^{\mu_1+1} & y^{\mu_1+1} \\ x^{\mu_2} & y^{\mu_2} \end{pmatrix} \Big/ \det\begin{pmatrix} x & y \\ 1 & 1 \end{pmatrix}$$

which is what we want at $z=1$. Now homogenize.

QED