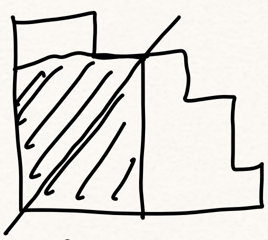
Rogers-Ramanujan Identities

① $$1+\sum_{k=1}^{\infty}\frac{q^{k^2}}{(1-q)(1-q^2)\cdots(1-q^k)}=\prod_{i=0}^{\infty}\frac{1}{(1-q^{5i+1})(1-q^{5i+4})}$$

② $$1+\sum_{k=1}^{\infty}\frac{q^{k(k+1)}}{(1-q)(1-q^2)\cdots(1-q^k)}=\prod_{i=0}^{\infty}\frac{1}{(1-q^{5i+2})(1-q^{5i+3})}$$

Lots of proofs; one modern combinatorial proof of ① involves the <u>Durfee square</u> of a partition:



largest square fitting inside λ.

<u>Ex:</u> Show

$$\sum p(n)x^n=\prod_{k=1}^{\infty}\frac{1}{1-x^k}$$

$$=\sum_{n=0}^{\infty}\frac{x^{n^2}}{\prod_{i=1}^{n}(1-x^i)^2}.$$

each partition on top and right of Durfee square

Such identities arise in analytic # theory; <u>modular forms</u>.

Ramanujan congruences and Dyson's rank

Thm: $p(5n+4) \equiv 0 \pmod 5$

Dyson's rank: width − height (mod 5)

rank: $4-1 = 3$ $3-2 = 1$ $2-2 = 0$ $2-3 \equiv -1 \equiv 4$ $1-4 = -3 \equiv 2$

Thm (Dyson): # partitions of $5n+4$ w/ rank $\equiv m \bmod 5$ is $\frac{1}{5}p(5n+4)$ for all $m = 0, 1, 2, 3, 4$.

Combinatorial proof?. Open.

Garsia-Milne Involution Principle (Sagan 2.3)

Garsia-Milne: gave an elem. proof of Rogers-Raman. using this.

<u>Idea</u>: to find a bijection $A \to B$, extend A, B to larger sets S, T w/ sign reversing involutions

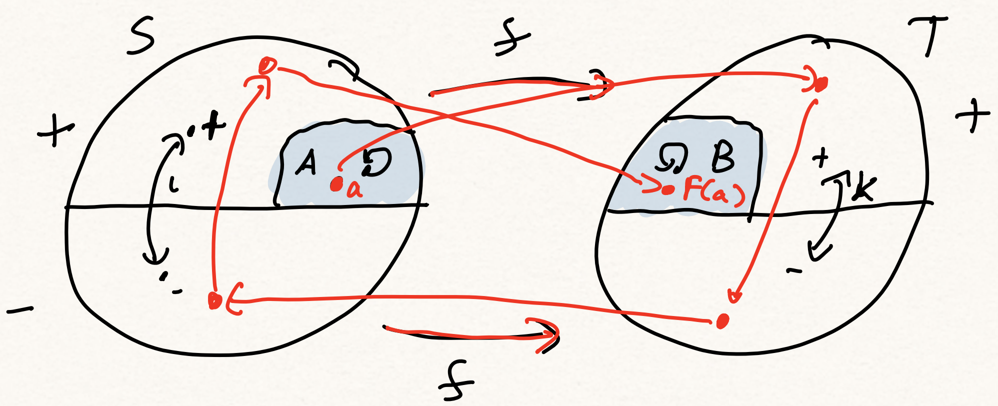
signed

$$\iota : S \to S$$
$$\kappa : T \to T$$

s.t. $\mathrm{Fix}\,\iota = A$ $\qquad$ (A, B positive signs)

$\mathrm{Fix}\,\kappa = B$.

Can construct a bijection $A \to B$ starting from <u>any</u> sign-preserving bijection $f : S \to T$.



Define $F: A \to B$ iteratively as follows:

1. If $f(a) \in B$ set $F(a) = f(a)$.
2. Otherwise, starting from $f(a) \in T$, apply
   κ, f^{-1}, ι, f.
   If the result lies in B, set $F(a)$ to be it.
3. Otherwise, repeat step 2 until we end in B.

Thm: The resulting function F is a well-defined bijection $F: A \to B$.

Pf: Consider the digraph D on $S \cup T$ given by:

- Arrows $f: S^+ \to T^+$
- Arrows $f^{-1}: T^- \to S^-$
- Arrows $\iota: S^- \to S^+$
- Arrows $\kappa: T^+ \to T^-$

Then elts of A have outdeg 1, indeg 0
from f

elts of B have indeg 1, outdeg 0
from f

Elts of $S-A$, $T-B$ each have indeg = outdeg = 1.

So we start in A and follow unique path in the arrows, always must terminate in B.

Thus F is well-defined.

F bijective: can reverse arrows, same process.

QED.

<u>Application</u>: $Q(n) = O(n)$

Let $A = \{$ partitions of n into distinct parts $\}$

$B = \{$ partitions of n into odd parts $\}$

Define $S = \{ (\lambda, I)$ s.t. $\lambda \vdash n$, $I \subseteq [n]$, part i occurs more than once in λ for all $i \in I$. $\}$

$T = \{ (\mu, I)$ s.t. $\mu \vdash n$, $I \subseteq [n]$, part $2i$ occurs in μ for all $i \in I$. $\}$

Note $A \subseteq S$ as $\{(\lambda, \emptyset) : \lambda \in A\}$
$B \subseteq T$ as $\{(\mu, \emptyset) : \mu \in B\}$

For elts of S and T, $\text{sgn}(\lambda, I) = (-1)^{|I|}$.

Sign-reversing involutions that fix A, B:

$$S \xrightarrow{\iota} S$$

$$(\lambda, I) \longmapsto \begin{cases} (\lambda, I \cup \{\text{max duplicate part } d \text{ in } \lambda\} & d \notin I \\ (\lambda, I \setminus \{\text{max } '' \quad '' \quad '' \quad ''\} & d \in I \\ (\lambda, I) & \lambda \text{ has dist. parts} \end{cases}$$

in this last case $I = \emptyset$ necessarily

$$T \xrightarrow{\kappa} T$$

$$(\mu, I) \longmapsto \begin{cases} (\mu, I \cup \{\text{max } i \text{ s.t. } 2i \in \mu\} & i \notin I \\ (\mu, I \setminus \{ '' \quad '' \quad '' \quad '' \} & i \in I \\ (\mu, I) & \mu \text{ odd parts} \end{cases}$$

$I = \emptyset$ in this last case.

- Sign-reversing. ✓
- A, B fixed sets w/ positive sign ✓.
- Now just need bijection:

$$f : S \to T$$
$$(\lambda, I) \longmapsto (\mu, I)$$

where μ formed by, for each $i \in I$, replacing one copy of i, i with $2i$. $\square$

<u>E.g.</u> 621

$$= (621, \emptyset) \xrightarrow{f} (621, \emptyset) \xrightarrow{\kappa} (621, 3) \xrightarrow{f^{-1}} (3321, 3) \xrightarrow{\iota} (3321, \emptyset)$$

$$\xrightarrow{f} (3321, \emptyset) \xrightarrow{\kappa} (3321, 1) \xrightarrow{f^{-1}} (33111, 1) \xrightarrow{\iota} (33111, 13)$$

$$\xrightarrow{f} (621, 13) \xrightarrow{\kappa} (621, 1) \xrightarrow{f^{-1}} (6111, 1) \xrightarrow{\iota} (6111, \emptyset)$$

$$\xrightarrow{f} (6111, \emptyset) \xrightarrow{\kappa} (6111, 3) \xrightarrow{f^{-1}} (33111, 3) \xrightarrow{\iota} (33111, \emptyset)$$

$$\xrightarrow{f} (33111, \emptyset).$$

$$= 33111$$