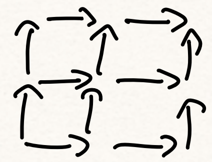
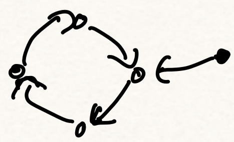
Lindström-Gessel-Viennot lemma

Def: Acyclic digraph - directed graph w/ no cycles
$D = (V, E)$

Ex: 

Ex: All Hasse diagrams of posets, orient up.

Non-ex: 

Setup: $a_1, \dots, a_n \in V$ sources, write $\underline{a}$ for short.
$b_1, \dots, b_n \in V$ sinks, write $\underline{b}$ for short.

- An n-path P from $\underline{a}$ to $\underline{b}$ is a collection of paths $a_i \longrightarrow b_{\pi(i)}$ in D for some permutation $\pi \in S_n$.
- The sign of P is $\mathrm{sgn}(\pi)$.
- An n-path is nonintersecting if none of the paths share vertices.
- Write $p(a_i, b_j)$ for the total # of paths from a_i to b_j.

Lemma (L-G-V): Let M be the matrix w/

$$M_{ij} = p(a_i, b_j).$$

Then $$\det(M) = \sum_{\substack{P \text{ nonintersecting} \\ n\text{-path}}} \operatorname{sgn}(P).$$

Pf: We first note that

$$\det(M) = \sum_{\pi \in S_n} \operatorname{sgn}(\pi)\, p(a_1, b_{\pi(1)})\, p(a_2, b_{\pi(2)}) \cdots p(a_n, b_{\pi(n)})$$

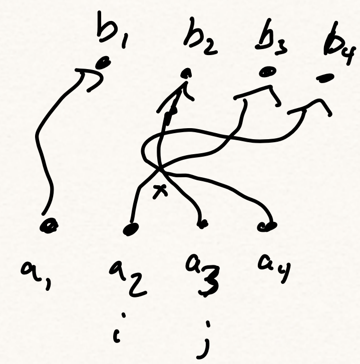
$$= \sum_{P \text{ } n\text{-path}} \operatorname{sgn}(P).$$

We construct a sign-reversing involution that cancels the intersecting n-path terms with each other.

Let P be an n-path w/ an intersection point. Let $a_i \xrightarrow{P_i} b_{\pi(i)}$ be the path w/ smallest index i that intersects another path.

Let x be first vertex along P_i intersecting another path.

Among all $p_j : a_j \to b_j$ that intersect p_i at x, let j be minimal. (Note $j > i$).



Involution: swap tails

$x \to b_{\pi(i)}$ with

$x \to b_{\pi(j)}$ in p_i, p_j.

First intersection point is preserved, so this is indeed an involution.

Also changes sign. ✓ $\square$

Note: When nonintersecting $\Rightarrow \pi = id$, the det counts the nonintersecting lattice paths.

Application: Cauchy-Binet

Thm: $\det MN = \sum_{K \in \binom{[n]}{m}} \det M|_K \cdot \det N|_K$

$M|_K$: cols, $N|_K$: rows

(for $M = m \times n$, $N = n \times m$)

Ex: $M = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 0 \end{bmatrix}$ $N = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 0 & 3 \end{bmatrix}$

$m=2,\ n=3$

$MN = \begin{bmatrix} 5 & 7 \\ 1 & 0 \end{bmatrix}$ $\det = -7$

$$\det\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \cdot \det\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} + \det\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \cdot \det\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} + \det\begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix} \cdot \det\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

$$= -1 + 0 + (-6) = -7 \quad \checkmark$$

<u>Proof via Gessel–Viennot:</u> 2 sources to 2 sinks but 3 in between:

$\leftarrow N$

$\leftarrow M$

This is an acyclic digraph; $(MN)_{ij}$ = # paths $a_i \to b_j$.

$$\det = \sum_{\substack{\text{nonint} \\ P\ \underline{a} \to \underline{b}}} \operatorname{sgn}(P)$$

Alternatively: pick which c's to pass through, do

$$\det(M|_K) \cdot \det(N|_K)$$ for each subset K of the c's.

Counts same thing. QED

(To finish the proof: we have shown the poly. identity holds for all pos. int. matrices, hence holds algebraically.)

Addendum: since this fact comes up a lot, let's prove it:

<u>Lemma</u>: Let $x_1, \dots, x_n$ be variables and suppose $f(x_1, \dots, x_n)$ and $g(x_1, \dots, x_n)$ are polynomials/$\mathbb{C}$ such that $f(a_1, \dots, a_n) = g(a_1, \dots, a_n)$ for all $(a_1, \dots, a_n) \in \mathbb{N}^n$.

Then $f = g$ as polynomials.

<u>Proof</u>: First note that by instead comparing $f-g$ to 0, we may assume $g = 0$.

Now, we show the claim by induction on n.

For $n=1$, if we have $f(a)=0$ for ∞ many $a \in \mathbb{C}$, then $f(x)=0$ because any other polynomial, of $\deg = d$, has at most d roots by the fundamental theorem of algebra.

Induction: Now assume it is true for $n-1$ variables. Then $f(x_1, \dots, x_{n-1}, 0)$ is a polynomial in $n-1$ variables that is 0 on all $(a_1, \dots, a_{n-1}) \in \mathbb{N}^{n-1}$, so is identically 0.

Thus the coeff of x_n^0 is 0, so f is divisible by x_n.

Setting $f^* = \frac{f}{x_n}$, we can do this again to see that the coeff of x_n^1 is 0, and so on.

Thus $f=0$ as a polynomial.

QED.