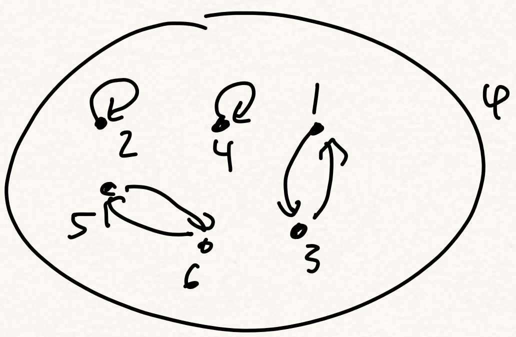
Sign-reversing involutions (Sagan 2.2)

Def: $\varphi: S \to S$ *involution* if $\varphi^2 = id$.

i.e. $\varphi = \varphi^{-1}$.

Involutions: fixed pts and 2-cycles only.



Def: A *signed set* is a set w/ an assignment of + or − to each elt.

Sign-reversing involution: Fixed pts are +, every 2-cycle has one + and one − (to cancel terms)

Ex: Show that $\sum (-1)^k \binom{n}{k} = 0$.

Sign-reversing involution on set of subsets:

$$A \mapsto A \cup \{n\} \text{ if } n \notin A$$

$$A \mapsto A - \{n\} \text{ if } n \in A.$$

No fixed pts. (sign is even/odd) $\square$

Ex: Prove that for $n \geq 0$,

$$\sum_{k \geq 0} (-1)^k k!\, S(n,k) = (-1)^n.$$

Note: $k!\, S(n,k) = \#$ ordered set partitions of $[n]$ into k blocks.

Sign function is $+$ if $\#$ blocks even
$-$ if $\#$ blocks odd.

Sign-reversing involution:

Ex for $n=3$:

even	odd
	123
12,3	1,2,3
13,2	1,3,2
23,1	2,1,3
1,23	2,3,1
2,13	3,1,2
3,12	3,2,1

Splitting: $\{a_1, a_2, \ldots, a_r\} \to \{a_1\}, \{a_2, \ldots, a_r\}$
if $r \geq 2$ $\uparrow$ min of block

Merging: $\{b\}, B \rightsquigarrow b \cup B$
if $b < \min B$

Involution: Find leftmost block that can either be split or merged w/ next block; perform that.

Ex: 4, 2, 15, 3 $\longleftrightarrow$ 4, 2, 1, 5, 3

split ... merge

4, 2, 35, 1 $\longleftrightarrow$ 4, 235, 1

merge ... split

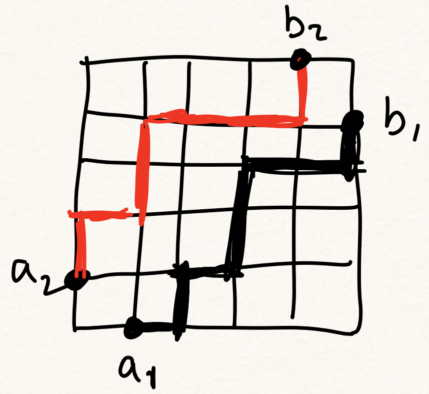
Can check this is an involution.

Only time it is fixed:

5, 4, 3, 2, 1. ✓

$\square$

Ex: How many pairs of paths on the grid from $(0,0)$ to $(5,5)$ do not intersect except at $(0,0)$ and $(5,5)$?



Reframe: want $a_1 \to a_2$, $b_1 \to b_2$ nonintersecting

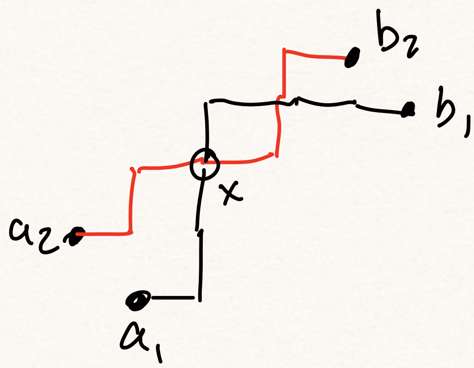
Sign-rev. involution method:
Count all pairs of paths from $\{a, b\}$ to $\{c, d\}$, possibly switching so that $a \to d$ and $b \to c$, w/ negative sign if so.

i.e. we compute

$$\sum_{P: \{a_1, a_2\} \to \{b_1, b_2\}} \operatorname{sgn}(P) \qquad (*)$$

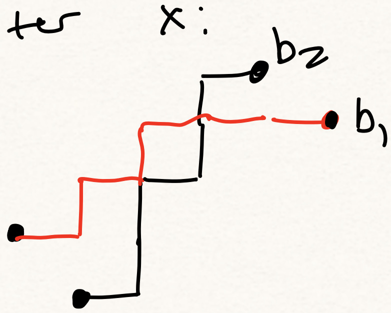
where $\operatorname{sgn}(P) = 1$ if $a_1 \to b_1$, $a_2 \to b_2$,
and $\operatorname{sgn}(P) = -1$ if $a_1 \to b_2$, $a_2 \to b_1$.

We first claim $(*)$ equals what we want to count; consider the sign-reversing involution that takes a pair of paths P and looks for their first point of intesection after a_1 on the path from a_1.



Call this point x; note it is also the first pt of intersection on the path from a_2 after a_2.

Define the involution by swapping the red and black tails of the paths after x: or fix if no such x.



This clearly is an involution and reverses the sign.

The fixed points are precisely the nonintersecting pairs of paths, so $(*)$ is what we are counting.

But we can count $(*)$ in another way:

\# $a_1 \to b_1$, $a_2 \to b_2$ paths

$$= \binom{8}{4}^2$$

\# $a_1 \to b_2$, $a_2 \to b_1$ paths

$$= \binom{8}{3} \cdot \binom{8}{5}$$

So
$$\text{ans} = \binom{8}{4}^2 - \binom{8}{3} \cdot \binom{8}{5}$$

$$= 70^2 - 56^2$$

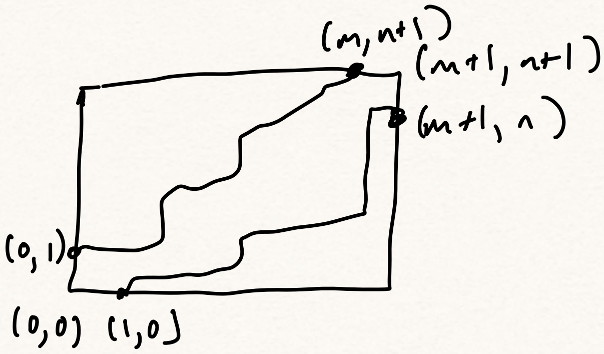
$$= 14 \cdot 126$$

$$= 1764$$

Note that the same argument shows:

<u>Prop</u>: \# pairs of nonintersecting lattice paths from $(0,0)$ to $(m+1, n+1)$ is

$$\binom{m+n}{n}^2 - \binom{m+n}{n+1}\binom{m+n}{n-1}.$$



Cor: $\binom{n}{k}^2 \geq \binom{n}{k-1}\binom{n}{k+1}$

for all k.

i.e. $\boxed{\binom{n}{k} \text{ log-concave!}}$

Def: A sequence is log-concave if $a_k^2 \geq a_{k-1} a_{k+1}$

for all k.

(i.e. log of the seq is concave)

Lemma: If a seq. of positive real #s $a_0, a_1, \ldots, a_n$ is log-concave, then it is unimodal -

$$a_0 \leq a_1 \leq \cdots \leq a_m \geq a_{m+1} \geq \cdots \geq a_n$$

for some m.

Pf: Assume not unimodal, so $\exists\, a_{k-1}, a_k, a_{k+1}$ with $a_{k-1} > a_k < a_{k+1}$. Then $a_k^2 < a_{k-1} a_{k+1}$, contradiction.

$\square$