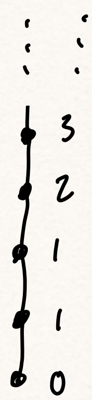
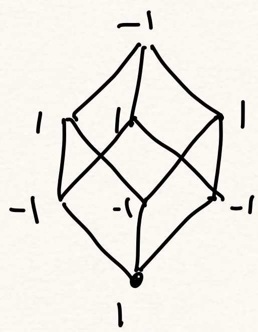
Intervals and the Incidence Algebra

<u>Sequence</u>: a function $f : (\mathbb{Z}_+, \leq) \to \mathbb{C}$



← Fibonacci sequence labeling the Hasse diagram of $(\mathbb{Z}_+, \leq)$

Now we consider more general functions on posets:



← Möbius function, used in inclusion-exclusion

More generally: Need functions on <u>intervals</u>.

<u>Def:</u> P is <u>locally finite</u> if every interval $[x, y]$ is finite.

← locally finite, not finite

<u>Def:</u> $\mathrm{Int}(P) = \{\text{intervals of } P\}$

Def: Let P locally finite. An *interval function* is a map $f: \mathrm{Int}(P) \to \mathbb{C}$, and its *interval generating function* is the formal sum

$$\sum_{x \le y} \underset{\substack{\uparrow \\ \text{coeff}}}{f(x,y)} \underset{\substack{\uparrow \\ \text{formal symbol}}}{[x,y]}$$

Incidence algebra:

$$\mathcal{I}(P) = \left(\{ f: \mathrm{Int}(P) \to \mathbb{C} \}, +, \cdot, \text{scaling} \right)$$

where:

- *Addition* $(+)$ is

$$\sum f(x,y)[x,y] + \sum g(x,y)[x,y]$$
$$= \sum \left(f(x,y) + g(x,y) \right) [x,y]$$

- *Multiplication* $(\cdot)$ is

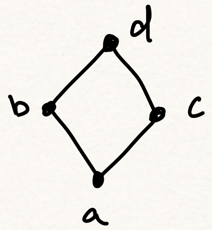
$$\left(\sum f(x,y)[x,y] \right) \left(\sum g(x,y)[x,y] \right)$$
$$= \sum_{x \le y} \left(\sum_{x \le z \le y} f(x,z) g(z,y) \right) [x,y].$$

i.e. $[x,z] \cdot [z,y] = [x,y]$,
$[x,w] \cdot [z,y] = 0$ if $w \ne z$.

- Scalar multiplication is

$$c \sum f(x,y)[x,y] = \sum c \cdot f(x,y)[x,y]$$

Ex:



$$f = [a,a] + 2[a,b] + [a,c]$$

$$g = -[a,d] + 3[b,d] + [c,c]$$

$$\Rightarrow f \cdot g = -[a,d] + 6[a,d] + [a,c] = 5[a,d] + [a,c]$$

Properties of an algebra:

① Addition is a group:

- Identity: $f(x,y) = 0$ for all $x \leq y$
- Inverses: negatives
- Associative ✓

② Mult. forms a ring w/ the addition:

- Associative: $[x,y] \cdot [y,w] \cdot [w,z] = [x,z]$ in any order; extend to general products

  → Note not commutative: $[x,y] \cdot [y,z] \neq 0 = [y,z] \cdot [x,y]$

- Identity: $\delta(x,y) = \begin{cases} 1 & x = y \\ 0 & \text{else} \end{cases}$

  Note: $f \cdot \delta = \delta \cdot f = f$ for all f

- Distributive: $f \cdot (g+h) = f \cdot g + f \cdot h$ by the def.

  Also $(g+h) \cdot f = g \cdot f + h \cdot f$

③ Scalar mult and addition form a vector space:

$$c(f+g) = cf + cg$$

$$c_1(c_2 f) = (c_1 \cdot c_2) f$$

④ Scalar mult compatible w/ $\bullet$

$$c(f \bullet g) = cf \bullet g = f \bullet cg.$$

Q: When do functions in the incidence algebra have mult. inverses?

A: When $f(x,x) \neq 0$ for all x (next hwk)
(If $f(x,x) = 0$ for some x, $f \cdot g(x,x) = \sum_{x \le z \le x} f(x,z) g(z,x) = 0$)

Ex: The zeta function of a poset:

$$\zeta(x,y) = 1 \text{ for all } x \le y$$

What is $\zeta \cdot \zeta$? ζ^k?

$$\zeta^2(x,y) = \sum_{x \le z \le y} 1 \cdot 1 = |[x,y]|$$

$$\zeta^k(x,y) = \sum_{x = x_0 \le x_1 \le \cdots \le x_k = y} 1 \cdot 1 \cdots 1 = \#\text{ "multichains" of length } k \text{ from } x \text{ to } y$$

$$(\zeta - \delta)^k(x,y) = \sum_{x = x_0 < x_1 < \cdots < x_k = y} 1 \cdot 1 \cdots 1 = \#\text{ chains length } k \text{ from } x \text{ to } y$$