Species isomorphism and Derangements

In class, we did an example that claimed

$$\mathcal{C}' = \mathcal{L} \quad \text{where}$$

$\mathcal{C}$ = species of cyclic permutations (necklaces)

$\mathcal{L}$ = species of perms. in list notation

We also said $\mathcal{R} = X(E \circ \mathcal{R})$.

What do these equalities of species mean?

<u>Def</u>: An <u>isomorphism</u> of species S, T is a collection of bijections

$$\varphi_A : S(A) \to T(A)$$

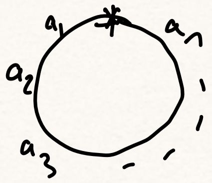
for each finite set A, that is compatible with relabeling: For any bijection $\pi : A \to B$, the diagram

$$\begin{tikzcd}
S(A) \arrow[r, "\varphi_A"] \arrow[d, "S_\pi"] & T(A) \arrow[d, "T_\pi"] \\
S(B) \arrow[r, "\varphi_B"] & T(B)
\end{tikzcd}$$

commutes. (To describe, suffices to find $\varphi_n = \varphi_{[n]}$'s)

Ex: Isomorphism $C' \cong L$:

$$\varphi_A : C'(A) \longrightarrow L(A)$$

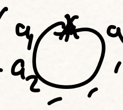
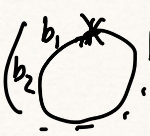


$\longmapsto a_1, a_2, a_3, \ldots, a_n$

is a bijection because we can insert the ✱ after a_n to clasp it back into a necklace.

Given a bijection $\pi : A \to B$
$\qquad a_i \mapsto b_i$

Have $L_\pi(\varphi_A($  $)) = L_\pi(a_1, \ldots, a_n) = b_1, \ldots, b_n$

$\varphi_B(C'_\pi($  $)) = \varphi_B($  $) = b_1, b_2, \ldots, b_n$

So $L_\pi \circ \varphi_A = \varphi_B \circ C'_\pi$,

and we have an isomorphism of species.

Derangements via species

Derangement: A permutation π s.t. $\pi_i \neq i$ for all i.

Ex: $2, 5, 4, 3, 1$
$\quad\ \ 1\ \ 2\ \ 3\ \ 4\ \ 5$

Cycle notation: no 1-cycles.

$(251)(34)$

Derangement: A permutation w/ no 1-cycles in cycle notation.

$\mathcal{D}$ = species of Derangements:

Note $\mathcal{D} = E \circ (\mathcal{C}_{2+})$

where $\mathcal{C}_{2+}$ is the species of cycles that have length at least 2;

since X is the indicator species that detects whether a set has size 1, we have $\mathcal{C}_{2+} = \mathcal{C} - X$.

So $\mathcal{D} = E \circ (\mathcal{C} - X)$

Now consider exp. gen fns: Recall

$$\tilde{\mathcal{C}}'(x) = \tilde{\mathcal{L}}(x) = \frac{1}{1-x}$$

$$\Rightarrow \tilde{\mathcal{C}}(x) = -\ln(1-x)$$

So
$$\tilde{\mathcal{D}}(x) = e^{\tilde{\mathcal{C}}(x) - x}$$
$$= e^{-\ln(1-x) - x}$$
$$= \frac{e^{-x}}{e^{\ln(1-x)}} = \boxed{\frac{e^{-x}}{1-x}}$$

Formula!

Corollary: $D_n = \#$ derangements

$$\sum \frac{D_n}{n!} x^n = \left(\sum \frac{(-1)^n}{n!} x^n\right)\left(\sum x^n\right)$$

$$= \sum_n \left(\sum_{k=0}^{n} \frac{(-1)^k}{k!}\right) x^n$$

So $\displaystyle \frac{D_n}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!} = \sum_{k=0}^{n} \frac{(-1)^{n-k}}{(n-k)!} \implies D_n = (-1)^n \sum_{k=0}^{n} (-1)^k (n)_k$

Ex: $D_4 = 1 - 4 + 4\cdot 3 - 4\cdot 3\cdot 2 + 4\cdot 3\cdot 2\cdot 1$

$= 12 - 4 + 1 = 9$