Proof of Cayley's formula using species

$\mathcal{T}$ = Species of labeled trees

R = Species of labeld rooted trees.

Note: $R = X \cdot \mathcal{T}'$

indicator species that sends $[1]$ to $\{[1]\}$, all else to $\emptyset$

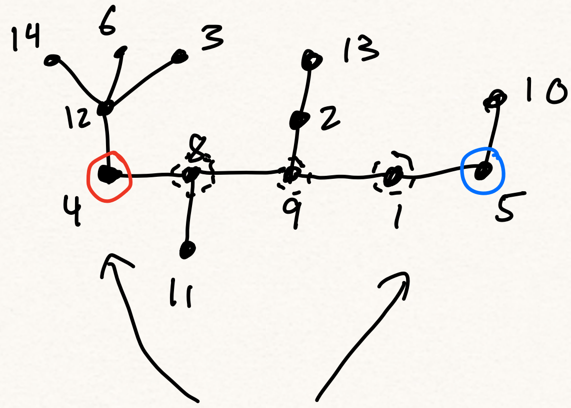
Want to show $\#$ trees $= n^{n-2}$

$\iff$ $\#$ rooted trees $= n^{n-1}$

$\iff$ $\#$ doubly rooted trees $= n^n$

Species of doubly rooted trees (same vertex can be rooted twice, or two different vertices):

$X \cdot R'$



two roots

Consider the path between the two roots, also circle each vertex along the path; each is a root of its branch, the doubly rooted tree can be thought of as a linear ordering on rooted trees

$\Rightarrow \quad \mathcal{L}^* \circ R = X \cdot R'$ $\qquad \mathcal{L}^* = \mathcal{L} - 1 = \frac{x}{1-x}$

$\Rightarrow \quad \frac{\tilde{R}(x)}{1-\tilde{R}(x)} = x\,\tilde{R}'(x)$

$$\frac{1}{1-\tilde{R}(x)} - 1 = x R'(x)$$

$$\frac{\tilde{R}(x)}{1-\tilde{R}(x)} = x R'(x)$$

$$\frac{1}{x} = R'(x)\left(\frac{1}{\tilde{R}(x)} - 1\right)$$

$$\ln(x) = \ln(\tilde{R}(x)) - \tilde{R}(x)$$

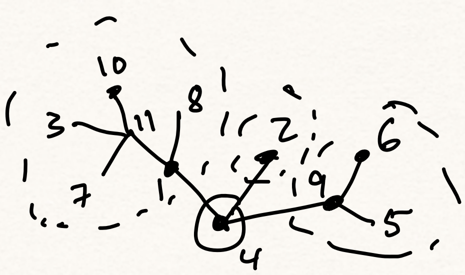
$$\ln\left(\frac{x}{\tilde{R}(x)}\right) = -\tilde{R}(x)$$

$$\frac{x}{\tilde{R}(x)} = e^{-\tilde{R}(x)}$$

$$\boxed{\tilde{R}(x) = x e^{\tilde{R}(x)}}$$

Alt method:

$R = X \cdot (E \circ R)$:



Lagrange Inversion Theorem

Suppose $F(G(x)) = x$. Then

$$g_n = \frac{1}{n}\left(\text{coeff of } x^{-1} \text{ in } \frac{1}{F(x)^n}\right)$$

Laurent series!

(Proof later).

Application ~~to rooted trees~~:

R is inverse to $F(y) = \frac{y}{e^y}$ since $x = \frac{R(x)}{e^{R(x)}}$.

So $$\frac{1}{n!}R_n = \frac{1}{n}\left(\text{coeff of } y^{-1} \text{ in } \frac{e^{ny}}{y^n}\right)$$

$$= \frac{1}{n}\left(\text{coeff of } y^{n-1} \text{ in } e^{ny}\right)$$

$$= \frac{1}{n}\left(\frac{n^n}{n!}\right) = \frac{1}{n!}\left(n^{n-1}\right).$$

QED

Proof of Lagrange Inversion (Stanley ch 5.4 vol 2)

Notation: $[x^n] f(x)$ = "coefficient of x^n in $f(x)$"

$$[x^4]\left(\frac{x}{1-x-x^2}\right) = F_4 = 3$$

<u>Lemma:</u> (i) For $h(x) \in \mathbb{C}((x))$, $[x^{-1}]h'(x) = 0.$

(ii) For $f(x) \in x\mathbb{C}[[x]]$ w/ $[x]f(x) \neq 0$,

$$[x^{-1}]f(x)^i f'(x) = \begin{cases} 1 & \text{if } i = -1 \\ 0 & \text{else} \end{cases}$$

<u>Pf:</u> (i) Clear.

(ii): $f(x)^i \cdot f'(x) = \frac{1}{i+1}\left(f(x)^{i+1}\right)'$

Now apply (i). If $i \neq -1$, it's 0.

If $i = -1$,

$$f(x)^i f'(x) = \frac{f'(x)}{f(x)} = \frac{a_1 + 2a_2 x + 3a_3 x^2 + \cdots}{a_1 x + a_2 x^2 + a_3 x^3 + \cdots}$$

$$= \frac{a_1 + 2a_2 x + 3a_3 x^2 + \cdots}{(a_1 x)(1 + \frac{a_2}{a_1} x + \cdots)}$$

$$= x^{-1} + \cdots$$

$\square$

Thm: $[x^n]g(x) = \frac{1}{n}[x^{-1}]\frac{1}{f(x)^n}$ if $f(g(x)) = x$

Pf: Suppose $g(x) = \sum b_i x^i$

$$x = g(f(x)) = \sum_{i\geq 1} b_i f(x)^i$$

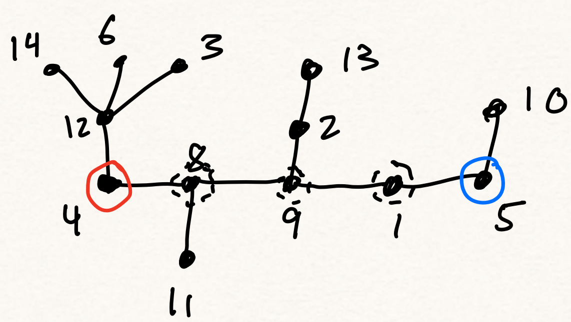
$$1 = \sum_{i\geq 1} i b_i f(x)^{i-1} f'(x)$$

$$\frac{1}{f(x)^n} = \sum_{i\geq 1} i b_i f(x)^{i-1-n} f'(x)$$

Coeff of x^{-1} $= n b_n$ by Lemma. ✓

QED

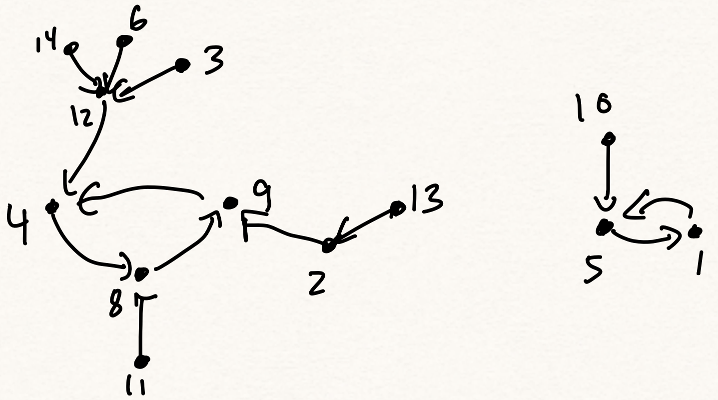
Bijective proof of Cayley



Show doubly rooted trees counted by n^n:

$n^n = \#$ functions $f: [n] \to [n]$.

Such a function's digraph looks like:



$(4\ 8\ 9)(1\ 5)$

i.e. all paths eventually end up in a cycle.

From the tree, orient all edges towards path btwn roots. For path between roots, use the bijection between line notation and cycle notation to sort them into cycles. $\square$