Species operations and EGF's

① **Addition of species** $S+T$:

$$(S+T)([n]) = S([n]) \sqcup T([n])$$

Relabeling:

$$(S+T)_\pi \cdot x = \begin{cases} S_\pi \cdot x & \text{if } x \in S([n]) \\ T_\pi \cdot x & \text{if } x \in T([n]) \end{cases}$$

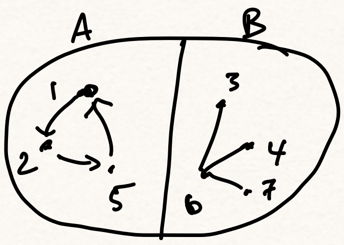
Note: EGF's add:

$$\tilde{S}(x) + \tilde{T}(x) = \widetilde{S+T}(x).$$

② **Multiplication**

$$S \cdot T([n]) = \left\{ (A, B, \alpha, \beta) : \begin{array}{l} A \sqcup B = [n], \\ \alpha \in S(A), \\ \beta \in T(B) \end{array} \right\}$$

Relabeling: $\underset{\substack{\| \\ (A,B,\alpha,\beta)}}{(S \cdot T)_\pi \cdot x} = (\pi A, \pi B, S_\pi \alpha, T_\pi \beta)$

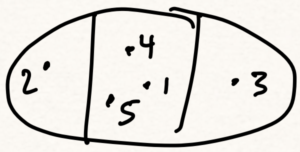


Ex: S = species of permutations
T = species of trees

ways to do this is $\sum \binom{n}{k} s_k t_{n-k}$

so $\widetilde{S \cdot T}(x) = \tilde{S}(x)\tilde{T}(x)$.

Ex: $E \cdot E \cdot E([n]) = \{(A, B, C)$ ordered set partition of n into 3 (possibly empty) blocks $\}$



Sanity check: $e^x \cdot e^x \cdot e^x = e^{3x} = \sum \frac{3^n}{n!} x^n$

3^n possibilities. ✓

③ Differentiation

$$S'([n]) = S([n] \cup \{*\})$$

"pointing"

$S'_\pi(X) = S_{\pi'}(x)$ where $\pi'(*) = *$

$\pi'(i) = \pi(i)$

Ex: $\mathcal{L}'$ = species of objs that look like → $152*43$

So $\mathcal{L}' = \mathcal{L}^2$

$\Rightarrow \dfrac{\tilde{\mathcal{L}}'(x)}{\tilde{\mathcal{L}}^2(x)} = 1 \quad \Rightarrow \dfrac{-1}{\tilde{\mathcal{L}}(x)} = x + c \qquad c = -1$

$\Rightarrow \tilde{\mathcal{L}}(x) = \dfrac{1}{1-x}$

Composition

F, G species w/ $G(\emptyset) = \emptyset$

$F \circ G([n]) = \{ (P, \underline{\alpha}, \beta)$ where

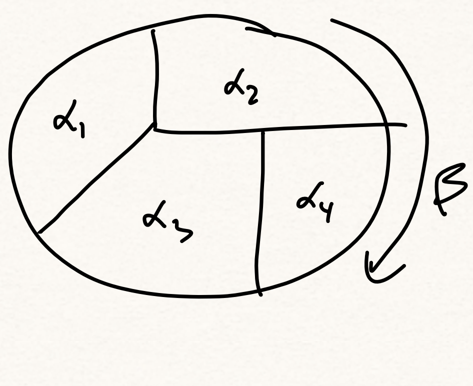
- P is a set partition $A_1 \sqcup A_2 \sqcup \cdots \sqcup A_k = [n]$
- $\underline{\alpha} = (\alpha_1, \dots, \alpha_k)$ consists of elts $\alpha_i \in G(A_i)$ $\forall i$
- $\beta \in F([k])$ $\}$

i.e.
- Set partition of $[n]$
- G-structures on each block
- F-structure on the set of blocks

Check that this corresp. to comp of EGF's:

$$\tilde{F} \circ \tilde{G}(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{\alpha_1 + \cdots + \alpha_k = n} \frac{1}{k!} \binom{n}{\alpha_1, \dots, \alpha_k} g_{\alpha_1} \cdots g_{\alpha_k} f_k \right) x^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{\substack{A_1 \sqcup \cdots \sqcup A_k \\ \text{set partition} \\ \text{of } [n] \\ |A_i| = \alpha_i}} g_{\alpha_1} \cdots g_{\alpha_k} f_k \right) x^n$$



✓

Ex: How many ways can you make a sequence of nonempty necklaces out of beads labeled $1, \ldots, n$ to display in a store window in order?

(Necklace = cyclic ordering of beads) ← G structure

(Sequence of necklaces) ← F structure

$\mathcal{F} = \mathcal{L}$

$G = \mathcal{C}$ ← species of cycles, nonempty

$$\tilde{\mathcal{C}}(x) = \sum_{n=1}^{\infty} \frac{(n-1)!}{n!} x^n = \sum_{n \geq 1} \frac{1}{n} x^n$$

$$= -\ln(1-x)$$

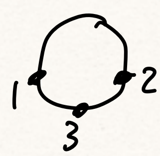
$$\mathcal{L} \circ \mathcal{C} = \frac{1}{1 + \ln(1-x)}$$

Sage answers:

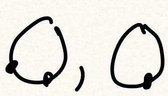
1, 1, 3, 14, 88, ...

two beads

three beads



$\times 2$



$\times 6$

$\times 6$

$6 + 6 + 2 = 14$ ✓

Ex: Bell numbers:

$$E \circ (E - 1) \quad \rightsquigarrow \quad e^{e^x - 1}$$

Ex: $E \circ \mathcal{C} = \mathcal{P}$