Exponential generating functions and species

Def: Exponential Generating Function (EGF) of $a_0, a_1, a_2, \ldots$ is

$$\tilde{A}(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n .$$

Advantage for labeled objects. Product:

$$\tilde{A}(x)\tilde{B}(x) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{a_k}{k!} \frac{b_{n-k}}{(n-k)!} \right) x^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} \right) x^n .$$

Ex: Bell numbers $B(n) = \#$ set partitions of $[n]$

$$= \sum_k S(n,k)$$

Recursion: $B(0) = 1$ and

$$B(n+1) = \underset{\substack{\uparrow \\ \text{n is in} \\ \text{its own} \\ \text{block}}}{B(n)} + \underset{\substack{\uparrow \\ \text{n shares} \\ \text{block w/} \\ \text{1 other}}}{n \cdot B(n-1)} + \underset{\substack{\uparrow \\ \text{n shares} \\ \text{block with} \\ \text{2 others}}}{\binom{n}{2} B(n-2)} + \cdots + \underset{\substack{\uparrow \\ \text{a single} \\ \text{block}}}{\binom{n}{n} B(0)}$$

$B(1) = 1$

$B(2) = B(1) + 1 \cdot B(0) = 2$

$B(3) = B(2) + 2 \cdot B(1) + B(0) = 5$

$$B(4) = B(3) + 3B(2) + 3B(1) + B(0) = 5 + 3 \cdot 2 + 3 + 1 = 15$$

$\vdots$

$1, 1, 2, 5, 15, \ldots$

<u>EGF:</u> $\tilde{B}(x) = \sum \frac{B(n)}{n!} x^n$

Note $B(n+1) = \sum_{k=0}^{n} \binom{n}{k} B(k)$

multiply by e^x

$$e^x \cdot \tilde{B}(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{k=0}^{n} \binom{n}{k} B(k) \right) x^n$$

$$= \sum_{n=0}^{\infty} \frac{B(n+1)}{n!} x^n$$

<u>Derivatives of EGFs:</u> $\frac{d}{dx} \tilde{B}(x) = \sum_{n=1}^{\infty} \frac{B(n)}{(n-1)!} x^{n-1} = \sum \frac{B(n+1)}{n!} x^n$

$$\Rightarrow \quad e^x \tilde{B}(x) = \frac{d}{dx} \tilde{B}(x)$$

Solving a differential eqn: put all $\tilde{B}$'s on one side:

$$e^x = \frac{\frac{d}{dx}\tilde{B}(x)}{\tilde{B}(x)}$$

Integrate:

$e^x + c = \ln(\tilde{B}(x))$ for some c

$\tilde{B}(x) = e^{e^x + c}$ What is c?

$\tilde{B}(0) = 1 \Rightarrow e^{1+c} = 1 \Rightarrow c = -1$

$$\Rightarrow \boxed{\tilde{B}(x) = e^{e^x - 1}}$$

$$= \sum \frac{1}{n!}(e^x - 1)^n$$

$$= \sum \frac{1}{n!}\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)^n$$

$F \circ G(x)$:
$$\sum_k \sum_{\alpha_1 + \cdots + \alpha_k = n} f_k \, g_{\alpha_1} g_{\alpha_2} \cdots g_{\alpha_k}$$

Coeff of $\frac{x^n}{n!}$ is $\displaystyle\sum_k \sum_{\alpha_1 + \cdots + \alpha_k = n} \frac{1}{k!} \frac{n!}{\alpha_1! \alpha_2! \cdots \alpha_k!}$

$$= \sum_k \sum_{\substack{\alpha_1 + \cdots + \alpha_k = n \\ \alpha_i \text{ nonzero}}} \frac{1}{k!} \binom{n}{\alpha_1, \alpha_2, \ldots, \alpha_k} \checkmark$$

unordered set partitions

Composition of EGF's: If $g_0 = 0$,

$$\tilde{F}(\tilde{G}(x)) = \sum_{n=0}^{\infty} \left(\sum_k \sum_{\alpha_1 + \cdots + \alpha_k = n} \frac{f_k}{k!} \frac{g_{\alpha_1}}{\alpha_1!} \cdots \frac{g_{\alpha_k}}{\alpha_k!} \right) x^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{\alpha_1 + \cdots + \alpha_k = n} \frac{1}{k!} \underbrace{\binom{n}{\alpha_1 \ldots \alpha_k}}_{\text{set partition the labels}} g_{\alpha_1} \cdots g_{\alpha_k} \cdot f_k \right) x^n$$

assign a G combinatorial obj to each block

structure the blocks as an f comb. obj.

Theory of species/labeled structures (Sagan ch 4)

Def: A set of labeled objects is a set w/ an action of S_n: a map

$$S_n \times A \to A \qquad \text{s.t.} \qquad \pi \cdot \sigma \cdot x = (\pi \circ \sigma) \cdot x$$
$$(\pi, x) \mapsto \pi \cdot x$$

The action is also called relabeling and we draw the elts of A to represent it as relabeling.

Ex: Say $A = \binom{[n]}{k}$, for inst. $A = \binom{[4]}{2}$.

has action of S_4: Write

$$S_4 \curvearrowright \{\underline{12}, \underline{13}, \underline{14}, \underline{23}, \underline{24}, \underline{34}\}$$

$$(12) \cdot \underline{14} = \underline{24} \qquad \text{etc.}$$

Ex: Say $A = S_n$. Two natural actions:

① $S_n \curvearrowright S_n$ by left multiplication:

$$\pi \cdot \sigma = \pi \circ \sigma$$

This is relabeling in list notation:

$$(12) \cdot 34152 = 34251$$

② $S_n \curvearrowright S_n$ by conjugation:

$$\pi \cdot \sigma = \pi \sigma \pi$$

This is relabeling in cycle notation:

$$(132) \circ (15)(234) \circ (123)$$

$$= (35)(124)$$

Def 1 (less formal) A <u>species</u> consists of

① An <u>assignment</u> to each integer n a set $S([n])$

② A rule for <u>relabeling</u>: For any $\pi \in S_n$, a map $S_\pi : S([n]) \to S([n])$

s.t. if $\sigma \in S_n$, $S_\sigma \circ S_\pi = S(\sigma \circ \pi)$

and $S(id) = id$.

(i.e. action for each n)

Def 2 (formal, allows labels besides $[n]$)

A <u>species</u> is a functor S from the category N to itself where N is:

<u>obj</u> = Sets

<u>mor</u> = bijections.

So have sets $S(A)$ and relabeling rules

$$S(A) \to S([n]) \quad \text{if } A \cong [n].$$

Examples

① $\mathcal{L}$ = species of permutations in list notation

② $\mathcal{P}$ = " " " in cycle notation

(note 1,2 are different b/c of different relabeling rules).

→ Two species are equivalent if there is an invertible <u>natural</u> <u>transformation</u> between them:

$$\begin{tikzcd} N \arrow[d, "S"'] \arrow[r, phantom, "\overset{\cong}{\eta}"] & N \arrow[d, "T"] \\ N & N \end{tikzcd}$$

η is: for every $A \in N$, a bijection

$$\eta_A : S(A) \to T(A)$$

s.t. these squares commute:

$$\begin{tikzcd} S(A) \arrow[r, "\eta_A"] \arrow[d, "S_\pi"'] & T(A) \arrow[d, "T_\pi"] \\ S(B) \arrow[r, "\eta_B"] & T(B) \end{tikzcd} \qquad \text{for all } \pi : A \to B$$

Proof that $\mathcal{P}$ and $\mathcal{L}$ are distinct:

If there were a natural transformation, $f = \eta_{[n]}$, then

$$\begin{tikzcd} \mathcal{P}([n]) \arrow[r, "f"] \arrow[d, "\mathcal{P}_\pi"] & \mathcal{L}([n]) \arrow[d, "\mathcal{L}\pi"] \\ \mathcal{P}([n]) \arrow[r, "f"] & \mathcal{L}([n]) \end{tikzcd}$$

commutes

which can be drawn

$$\begin{tikzcd} \mathcal{P}([n]) \arrow[loop left, "\mathcal{P}_\pi"] \arrow[r, "f"] & \mathcal{L}([n]) \arrow[loop right, "\mathcal{L}\pi"] \end{tikzcd}$$

for all π. In particular f sends the S_n action to an isomorphic action on $\mathcal{L}([n])$. But these actions have different orbit sizes, so no such f exists. QED.

Other ex's:

(3) E = "trivial species"

$E([n]) = \{[n]\}$ for all n, trivial action

(4) $\mathbb{1}$ = "indicator species":

$$\mathbb{1}([n]) = \begin{cases} \{\emptyset\} & \text{if } n=0 \\ \{\} & \text{else} \end{cases}$$

⑤ $\mathcal{T}$ = "species of labeled trees"

$\mathcal{T}(\{n\})$ = {labeled trees on n vertices}

relabel as drawn.

⑥ $\mathcal{B}$ = species of set partitions.

etc.