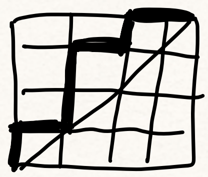
Catalan numbers

① **Dyck paths:** How many paths from $(0,0)$ to (n,n) stay weakly above the diagonal?



$n=0$: 1

$n=1$: 1 

$n=2$: 2 

$n=3$: 5

Sequence: $1, 1, 2, 5, 14, 42, 132, \ldots$
 $\| $
$C_0, C_1, C_2, C_3, \ldots$

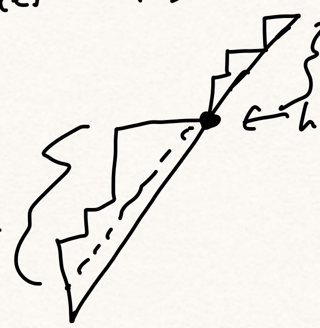
Catalan numbers

Thm: $C_n = \frac{1}{n+1}\binom{2n}{n}$ (proof later)

Stanley: 214 interpretations!

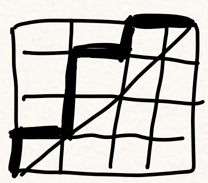
Recursion: $C_{n+1} = C_0 C_n + C_1 C_{n-1} + C_2 C_{n-2} + \cdots + C_n C_0$

Pf: Consider first return to diagonal:



Other interpretations

② **Ballot sequences:**



$\longleftrightarrow$ $\underbrace{\text{URUURURR}}$

sequence of up's and Rights corresp. to a Dyck path if as we read left to right, #R's never exceeds #U's.

Q: People line up to vote for U or R. An equal number will vote for each. Probability that R is never ahead?

$\frac{1}{n+1}$

So $C_n = $ # Dyck paths $=$ # ballot words.

③ Parenthesizations of products:

$\boxed{\begin{matrix}(a)\\ 1\end{matrix}}$ $\boxed{\begin{matrix}(a\cdot b)\\ 1\end{matrix}}$ $\boxed{\begin{matrix}(a\cdot b)\cdot c = a\cdot(b\cdot c)\\ 2\end{matrix}}$

$\boxed{\begin{matrix}(a\cdot b)\cdot(c\cdot d),\ ((a\cdot b)\cdot c)\cdot d,\ (a\cdot(b\cdot c))\cdot d,\ a\cdot((b\cdot c)\cdot d),\ a(b\cdot(c\cdot d))\\ 5\end{matrix}}$

ways to parenthesize a product w/ n $\cdot$ symbols is C_n

<u>Pf:</u> Show it satisfies the recursion:

Consider the top level product:

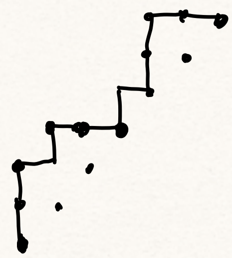
$$\underbrace{(a\cdot b\cdot c)}_{\substack{C_2\\ \text{ways to}\\ \text{parenthesize}}}\cdot\underbrace{(d\cdot e\cdot f\cdot g)}_{\substack{C_3\\ \text{ways}}}$$

$$\Rightarrow C_6 = C_0C_5 + C_1C_4 + C_2C_3 + C_3C_2 + C_4C_1 + C_5C_0$$

Can unwind the recursion into a bijection: last product $\Leftrightarrow$ 1st return to diag.

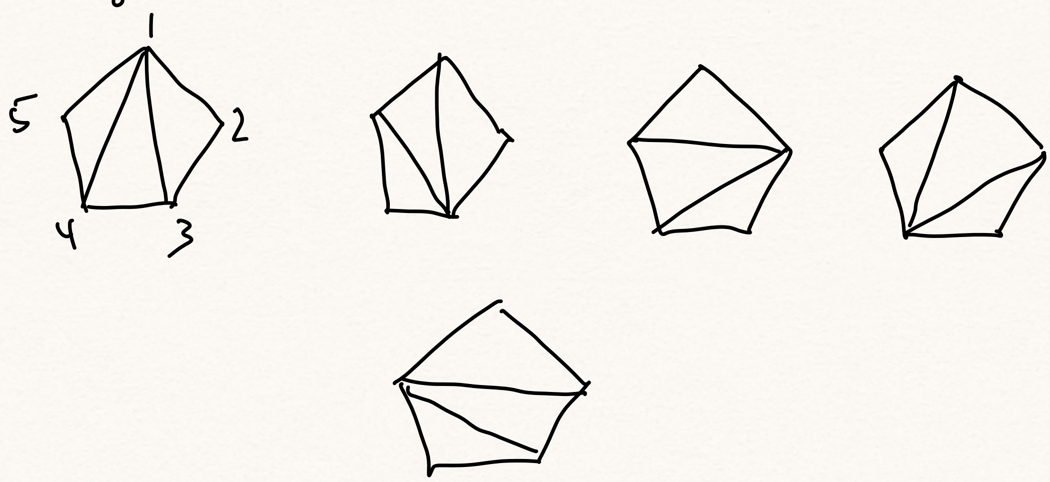
$$(a \cdot (b \cdot c)) \cdot (d((e \cdot f) \cdot g))$$

$$\updownarrow$$



$$C_2 = C_0 C_1 + C_1 C_0$$

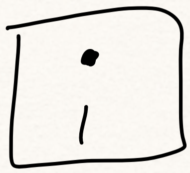
④ Triangulations of a labeled $(n+2)$-gon

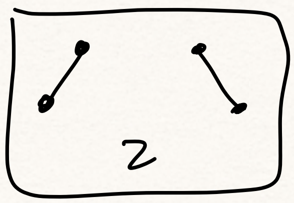


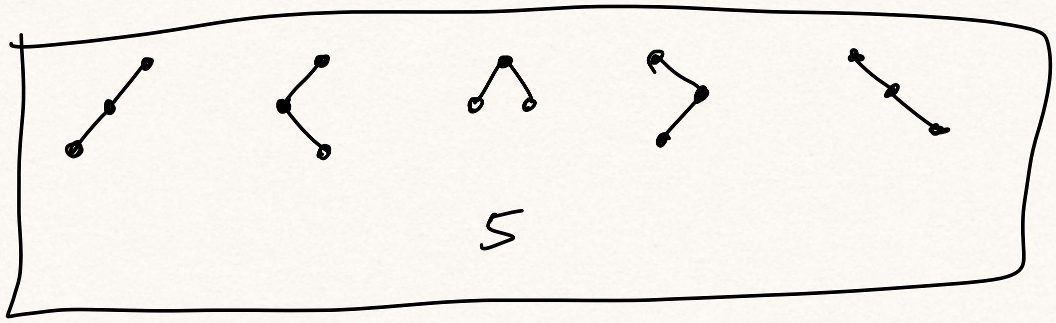
⑤ Tableaux, $2 \times n$, rows + cols increaing

$$\boxed{\begin{array}{ccc} 4 & 5 & 6 \\ 1 & 2 & 3 \end{array}}, \quad \begin{array}{ccc} 3 & 5 & 6 \\ 1 & 2 & 4 \end{array}, \quad \begin{array}{ccc} 3 & 4 & 6 \\ 1 & 2 & 5 \end{array}, \quad \begin{array}{ccc} 2 & 5 & 6 \\ 1 & 3 & 4 \end{array}, \quad \begin{array}{ccc} 2 & 4 & 6 \\ 1 & 3 & 5 \end{array}$$

⑥ **Binary trees**: Start w/ root, each node has left, right, both, or neither children







(Easy bijection with parenthesizing)

Generating function and proof of explicit formula

Recall: $C_0 = 1$ and

$$C_{n+1} = C_0 C_n + C_1 C_{n-1} + C_2 C_{n-2} + \cdots + C_n C_0$$

$$= \sum_{k=0}^{n} C_k C_{n-k}$$

Note: If $C(x) = \sum_{n=0}^{\infty} C_n x^n$,

$$C(x)^2 = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} C_k C_{n-k} \right) x^n$$

$$= \sum_{n=0}^{\infty} C_{n+1} x^n = \frac{1}{x} \sum_{n=0}^{\infty} C_{n+1} x^{n+1}$$

$$= \frac{1}{x} \sum_{n=1}^{\infty} C_n x^n$$

$$= \frac{1}{x}\left(C(x) - 1 \right).$$

So $x\,C(x)^2 = C(x) - 1$

$$x\,C(x)^2 - C(x) + 1 = 0$$

Solve for $C(x)$ w/ quadratic formula:

$$\boxed{C(x) = \frac{1 \pm \sqrt{1-4x}}{2x}.}$$

Plus or minus? Need $2x$ to divide into top, $\sqrt{1-4x}$ as Taylor series is

$$1 - 2x - 2x^2 - \cdots$$

So it's minus.

$$\Rightarrow \quad \boxed{C(x) = \frac{1 - \sqrt{1-4x}}{2x}}$$

Expanding another way:

Generalized Binomial theorem says

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k \quad \text{where } \binom{\alpha}{k} = \frac{(\alpha)_k}{k!}.$$

(for any real number α!)

Pf: If a Taylor series expansion identity holds on a nontrivial domain, then the identity holds as generating functions,

and we have, if $f(x) = (1+x)^\alpha$:

$$f(0) = 1$$

$$f'(x) = \alpha(1+x)^{\alpha-1} \Rightarrow f'(0) = \alpha$$

$$f''(x) = \alpha(\alpha-1)(1+x)^{\alpha-2} \Rightarrow \frac{f''(0)}{2!} = \frac{\alpha(\alpha-1)}{2!}$$

$$\vdots$$

$$f^{(k)}(x) = (\alpha)_k (1+x)^{\alpha-k} \Rightarrow \frac{f^{(k)}(0)}{k!} = \frac{(\alpha)_k}{k!}.$$

QED

Now: $$C(x) = \frac{1-\sqrt{1-4x}}{2x} = \frac{1 - \sum_{k=0}^{\infty} \binom{1/2}{k}(-4)^k x^k}{2x}$$

$$= \frac{1}{2x}\left(\sum_{k=1}^{\infty} - \frac{(1/2)_k}{k!}(-4)^k x^k\right.$$

$\frac{1}{2}-k+1$

$$= \frac{1}{2x}\sum_{k=1}^{\infty} -\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{(2k-3)}{2}\right)}{k!}(-1)^k 2^{2k} x^k$$

$$= \frac{1}{2x}\sum_{k=1}^{\infty} \frac{1}{2^k}\,\frac{1\cdot 3\cdot 5\cdots(2k-3)}{k!}\cdot 2^{2k} x^k$$

$$= \frac{1}{\cancel{2}x}\sum_{k=1}^{\infty} \frac{1\cdot 3\cdot 5\cdot\ldots\cdot(2k-3)\cdot 2\cdot 4\cdot\ldots\cdot(2k-2)}{k!\cdot(2\cdot 4\cdot\ldots\cdot 2k-2)}\cdot 2^{k-1} x^k$$

$$= \cancel{\frac{1}{x}}\sum_{k=1}^{\infty} \frac{(2k-2)!}{k!\,(k-1)!} x^{k-1}$$

$$= \sum_{k=0}^{\infty} \frac{(2k)!}{(k+1)!\,k!} x^k$$

$$= \sum_{k=0}^{\infty} \frac{1}{k+1}\binom{2k}{k} x^k.$$

QED.

Recall Stanley problem: Prove

$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2(n-k)}{n-k} = 4^n;$$

Want to show $\displaystyle\frac{1}{1-4x} = \left(\sum_{n=0}^{\infty}\binom{2n}{n}x^n\right)^2$

i.e. that $\displaystyle(1-4x)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\binom{2n}{n}x^n$

Take $\frac{d}{dx}(xC(x))$:

$$xC(x) = \frac{1-\sqrt{1-4x}}{2} = \sum_{k=0}^{\infty} \frac{1}{k+1}\binom{2k}{k}x^{k+1}$$

$$\frac{d}{dx}(xC(x)) = \cancel{-\frac{1}{2}\left(\frac{1}{2}\right)(-4)}(1-4x)^{-\frac{1}{2}} = \sum_{k=0}^{\infty}\binom{2k}{k}x^k$$

$$\Rightarrow (1-4x)^{-\frac{1}{2}} = \sum \binom{2n}{n}x^n$$

as desired. QED.

Bijective proof of Catalan formula:

Note: $\frac{1}{n+1}\binom{2n}{n} = \frac{1}{2n+1}\binom{2n+1}{n}$.

We'll show the latter is C_n.

Note: $C_n = \#$ strictly ballot sequences of $n+1$ 0's, n 1's

(by adding a 0 to the start of a ballot sequence:

$C_3 = 5$:

0000111
0001011
0010011
0010101
0001101

Then, $\binom{2n+1}{n} = \#$ sequences w/ $(n+1)$ 0's, n 1's.

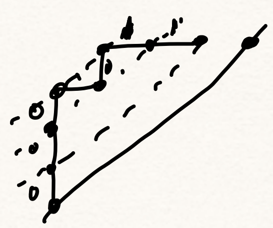
For each strictly ballot one, consider all $2n$ cyclic shifts:

$$\underbrace{0001011}_{\text{str. ballot}} \to \underbrace{0010110}_{\text{tie}} \to \underbrace{0101100}_{\text{tie}} \to \underbrace{1011000}_{\text{not ballot}}$$

$$\to \underbrace{0110001}_{\text{not ballot}} \to \underbrace{1100010}_{\text{not ballot}} \to \underbrace{1000101}_{\text{not ballot.}}$$

<u>Claim</u>: All $2n$ cyclic shifts are not ballot:

If 0 is up, 1 is right, there is a unique lowest diagonal. By drawing other diagonals,



we see every other cyclic shift returns to diagonal.

Similarly each non-ballot word has a unique strictly ballot cyclic shift,

QED.