Permutation statistics and q-analogs

Recall: Plot notation

4, 2, 5, 1, 3

Descent: An index i s.t. $\pi_i > \pi_{i+1}$

Ascent: An index i s.t. $\pi_i < \pi_{i+1}$

Descents above: 1, 3
Ascents: 2, 4

Inversion: A pair (π_i, π_j) with $i<j$, $\pi_i > \pi_j$

Inversions above:

42, 41, 43, 21, 51, 53

Def: $\mathrm{inv}(\pi)$ = # inversions of π
$\mathrm{des}(\pi)$ = # descents of π
$\mathrm{maj}(\pi)$ = sum of the descents of π

These are all combinatorial statistics on S_n

$\pi \in S_3$	inv	maj	des
123	0	0	0
132	1	2	1
213	1	1	1
231	2	2	1
312	2	1	1
321	3	3	2

<u>Note:</u> inv and maj are <u>equidistributed</u> on S_3 - same number of permutations with each output value.

<u>Def:</u> A <u>combinatorial statistic</u> or <u>weight function</u> is a map

$$wt: A \to \mathbb{N}$$

where A is a set of combinatorial objects.

e.g. inv, maj, des are all combinatorial objects on S_n for all n.

<u>Def:</u> Two comb. statistics

$$wt_1 : A \to \mathbb{N}$$
$$wt_2 : B \to \mathbb{N}$$

are <u>equidistributed</u> if there is a bijection $f : A \to B$ s.t. $wt_2(f(a)) = wt_1(a)$ for all $a \in A$.

$$\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, "wt_1"] \arrow[d, phantom, shift left=4ex, "\circlearrowleft"] & \\ B \arrow[r, "wt_2"'] & \mathbb{N} \end{tikzcd}$$

Weighted counting

Def: Given a comb. stat $wt : A \to \mathbb{N}$, the <u>q-count</u> of A with respect to wt, a.k.a. the wt-<u>q-analog</u> of $|A|$, is

$$|A|_q := \sum_{a \in A} q^{wt(a)}.$$

Note: at $q=1$ we get $|A|$.

at $q=0$ we get $\#\{a \in A \text{ with } wt(a)=0\}$

$\boxed{\text{Note: } A, B \text{ equidistributed iff } |A|_q = |B_q|}$ ☆

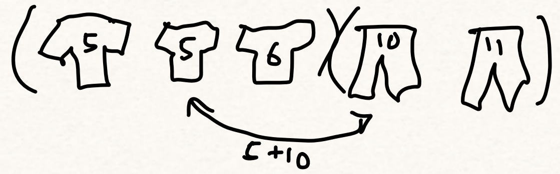
Why weighted counting

Q: We have 3 shirts weighing 5 oz, 5 oz, 6 oz and 2 pants weighing 10 oz, 11 oz.

a) How many outfits? $2 \cdot 3 = 6$

b) How many outfits of each total weight?

$$(q^5+q^5+q^6)(q^{10}+q^{11}) = q^{5+10}+q^{5+10}+q^{6+10}$$
$$\underbrace{+q^{5+11}+q^{5+11}+q^{6+11}}_{\text{one term for each outfit w/ weight}}$$



$$(2q^5+q^6)(q^{10}+q^{11}) = 2q^{15}+3q^{16}+q^{17}$$

So: 2 outfits weighing 15 oz

3 outfits weighing 16 oz

1 outfit weighing 17 oz.

$\boxed{\begin{array}{l} \underline{\text{Fact}}: |A|_q \cdot |B|_q = |A \times B|_q \quad \text{where if} \\ wt_1 : A \to \mathbb{Z} \text{ is } A\text{'s stat and} \\ wt_2 : B \to \mathbb{Z} \text{ is } B\text{'s stat,} \\ \\ wt : A \times B \to \mathbb{Z} \text{ is defined by} \\ \quad wt(a,b) = wt_1(a) + wt_2(b). \end{array}}$

<u>q-analog for inv</u>

In S_3 ex above: for <u>inv</u>:

$$|S_3|_q = 1+2q+2q^2+q^3 = (1+q)(1+q+q^2)$$

$$|S_4|_q = |S_3|_q(1+q+q^2+q^3) = 1+3q+5q^2+6q^3+5q^4 + 3q^5+q^6$$

	inv		inv		inv		inv
$123\underline{4}$	0	$12\underline{4}3$	1	$1\underline{4}23$	2	$\underline{4}123$	3
$132\underline{4}$	1	$13\underline{4}2$	2	$1\underline{4}32$	3	$\underline{4}132$	4
$213\underline{4}$	1	$21\underline{4}3$	2	$2\underline{4}13$	3	$\underline{4}213$	4
$231\underline{4}$	2	$23\underline{4}1$	3	$2\underline{4}31$	4	$\underline{4}231$	5
$312\underline{4}$	2	$31\underline{4}2$	3	$3\underline{4}12$	4	$\underline{4}312$	5
$321\underline{4}$	3	$32\underline{4}1$	4	$3\underline{4}21$	5	$\underline{4}321$	6

By this pattern:

<u>Thm</u>: $|S_n|_q = \sum_{\pi \in S_n} q^{inv(\pi)} = (1)(1+q)(1+q+q^2)\cdots(1+q+q^2+\cdots+q^{n-1})$

$$=: (n)_q!$$

"q-analog of n": $1+q+q^2+\cdots+q^{n-1} =: (n)_q$

"q-analog of n!": $(1)_q(2)_q(3)_q\cdots(n)_q =: (n)_q!$

<u>Thm</u>: (inv, maj equidistributed) We have

$$\sum_{\pi \in S_n} q^{maj(\pi)} = (n)_q! = \sum_{\pi \in S_n} q^{inv(\pi)}$$

(Notation: Any statistic on permutations that is equidistributed w/ inv is a <u>Mahonian</u> statistic).

<u>Pf 1</u>: Carlitz bijection: Use an intermediate object.

<u>Lehmer codes</u>:

$$L_n = \{(c_1, c_2, \ldots, c_n) : c_i \le i-1 \quad \forall i\}$$

<u>Note</u>: $|L_n| = n!$, $\quad |L_n|_q = \sum_{\underline{c} \in L_n} q^{\Sigma c_i} = (n)_q!$

(prove in class)

Will construct maps invcode, majcode s.t.

$$\begin{tikzcd}
S_n \arrow[d, "\text{invcode}"'] \arrow[dr, "\text{inv}"] & \\
L_n \arrow[r, "\Sigma"] & \mathbb{Z} \\
S_n \arrow[u, "\text{majcode}"] \arrow[ur, "\text{maj}"'] &
\end{tikzcd}$$

commutes:

$\text{invcode}(\pi) = (i_1, i_2, \ldots, i_n)$ where i_k = # inversions that k is the larger elt of

$$42513 \xrightarrow{\text{invcode}} (\underset{\le 0}{0}, \underset{\le 1}{1}, \underset{\le 2}{0}, \underset{\le 3}{3}, \underset{\le 4}{2})$$

Reverse map: insert $1, 2, 3, \ldots$ one at a time according to code.

<u>majcode</u>: $\pi|_i$ = remove $i+1, i+2, \ldots, n$ from π.

$$c_i = \text{maj}(\pi|_i) - \text{maj}(\pi|_{i-1})$$

<u>Ex</u>: $\text{majcode}(25143) = (0, 1, 0, 3, 2)$

$$\pi|_5 = 2\underline{5}1\underline{4}3 \qquad \text{maj} = 6$$
$$\pi|_4 = \underline{2}1\underline{4}3 \qquad \text{maj} = 4$$
$$\pi|_3 = 213 \qquad \text{maj} = 1$$
$$\pi|_2 = 21 \qquad \text{maj} = 1$$
$$\pi|_1 = 1 \qquad \text{maj} = 0$$

${}_2 2 {}_1 1 {}_3 3 {}_0$

${}_3 2 {}_2 1 {}_4 4 {}_1 3 {}_0$

Why a bijection: Given a perm of $n-1$, need to show that the n ways of inserting n increase the maj by a different value from 0 to $n-1$:

Ex: $\underline{3}\,1\,4\,\underline{6}\,\underline{5}\,2\,\underline{8}\,7$ ← insert 9

maj+: 4 3 2 1 0

Note that inserting n after the descents from R to L (incl. last entry) changes maj by $+0, +1, \ldots$. Then from L to R it increases by one more each time too. $\square$

Alternate: Foata (Stanley)

More q-analogs

Recall: $\binom{n}{k}$ = # binary sequences w/ k 0's, $n-k$ 1's $= |S_{0^k 1^{n-k}}|$

Def: $$\binom{n}{k}_q = \sum_{w \in S_{0^k 1^{n-k}}} q^{\mathrm{inv}(w)}$$

inv = # pairs consisting of a 1 to the left of a 0

Ex $\mathrm{inv}(1011001) = 7$

Thm: $\binom{n}{k}_q = \dfrac{(n)_q!}{(k)_q!\,(n-k)_q!}$

Pf: We'll prove $\binom{n}{k}_q (k)_q!\,(n-k)_q! = (n)_q!$

We have

$$\binom{n}{k}_q (k)_q!\,(n-k)_q! = \left(\sum_{w \in S_{0^k,1^{n-k}}} q^{\mathrm{inv}(w)}\right)\left(\sum_{\pi \in S_k} q^{\mathrm{inv}(\pi)}\right)\left(\sum_{\sigma \in S_{\{k+1,\dots,n\}}} q^{\mathrm{inv}(\sigma)}\right)$$

$$= \sum_{(w,\pi,\sigma)} q^{\mathrm{inv}(w)+\mathrm{inv}(\pi)+\mathrm{inv}(\sigma)}$$

For a given triple w, π, σ, construct $p \in S_n$ by replacing the 0's w/ π from L to R and the 1's w/ σ. Then since everything in σ is larger than everything in π, the inv's from w are retained, and we add on the inv's from π, σ.

Thus the above sum is

(reversible process by replacing $1,\dots,k$ w/ 0, $k+1,\dots,n$ w/ 1, extracting perms.)

$$= \sum_{p \in S_n} q^{\mathrm{inv}(p)} = (n)_q!$$