Permutation notations

<u>Def</u>: A <u>permutation</u> of a set A is a bijection $A \to A$. Most commonly use $[n] \to [n]$.

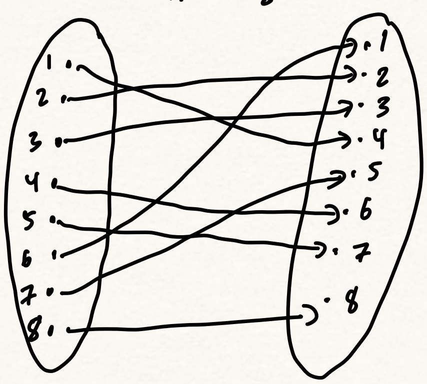
<u>Def</u>: $S_n = \{\text{permutations of } [n]\}$, $S_A = \{\text{perms of } A\}$

Symmetric group under composition $\circ$.

Ways to draw permutations

① Venn diagram:

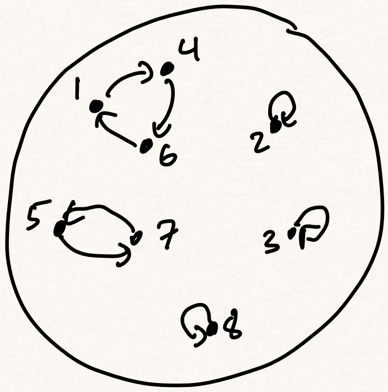
$\pi \in S_8$



② List notation:

$$4, 2, 3, 6, 7, 1, 5, 8$$
$$= \pi_1, \pi_2, \ldots, \pi_n$$
$$= \pi(1), \pi(2), \ldots, \pi(n)$$

③ Digraph



④ Cycle notation

$$(146)(2)(3)(57)(8)$$

$$= (146)(57)$$

omit fixed points

Why cycle notation is valid:

Lem: The digraph of every permutation is a union of disjoint cycles.

Pf: Every vertex has indeg = outdeg = 1.

Follow the unique edges forward starting from vertex x; if it loops back at $y \neq x$ then y has indeg > 1, $\rightarrow\leftarrow$.

So it loops back to x.

Now, start w/ some pt a not in the cycle; it also forms a loop disjoint from x's cycle, etc. $\square$

Composition: $\pi, \sigma \in S_n \Rightarrow \pi \circ \sigma \in S_n$

Computing $\pi \circ \sigma$ or any larger comp. in cycle notation:

Ex:

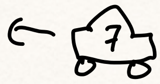
1. Write $\pi\sigma$ concatenated $\qquad \underbrace{(146)(57)}_{\pi}\,\underbrace{(13)(245)}_{\sigma}$
2. Open a cycle starting at 1 $\qquad (1 \ldots)$
3. To figure out what comes after 1 in its $\qquad (146)(57)(\underline{\underline{1}}5)(243)$

...

cycle, get in a car carrying 1 at the end of $\pi\sigma$ and drive left until we find a 1, and pick up the elt that 1 maps to. Now continue driving searching for the new elt, and so on. Whatever the car exits carrying is the image of 1.

$(146)(\underline{5}7)(15)(243)$

$(146)(57)(15)(243)$



(4) Write down this image after 1 in the cycle (if it's 1, close the cycle)

$(17\cdots)$

(5) Repeat the car procedure starting w/ the new elt until the cycle returns to 1.

(1754326)

(6) Choose a number that hasn't been used yet and open a new cycle and repeat.

$(1754326)(8)$

Note: This works for any product of cycles.

Note: We drop fixed points, so any π in cycle notation can be thought of as a product of its own cycles:

$$(13)(246)(57) = (13)\circ(246)\circ(57)$$

Stirling numbers of first kind

Def: $c(n,k)$ = # permutations of $[n]$ w/ k cycles (including fixed pts)

Ex: What is $c(4,2)$?

(123) (124) (134) (234)
(132) (142) (143) (243)

(12)(34) (13)(24) (14)(23)

11

Recursion: $c(n,n)=1$
$c(n,1) = (n-1)!$
$$c(n,k) = \underbrace{(n-1)c(n-1,k)}_{n \text{ not fixed}} + \underbrace{c(n-1,k-1)}_{n \text{ fixed}}$$

Stirling # of 1st kind:

$$s(n,k) := (-1)^{n-k} c(n,k)$$

Dual to second kind:

$s(n,k)$:

$n \backslash k$	1	2	3	4	5
1	1				
2	-1	1			
3	2	-3	1		
4	-6	11	-6	1	
5	24	-50	35	-10	1

$S(n,k)$:

$n \backslash k$	1	2	3	4	5
1	1				
2	1	1			
3	1	3	1		
4	1	7	6	1	
5	1	15	25	10	1

$$S(n,k) = S(n-1,k-1) + k\,S(n-1,k)$$

Inverse matrices:

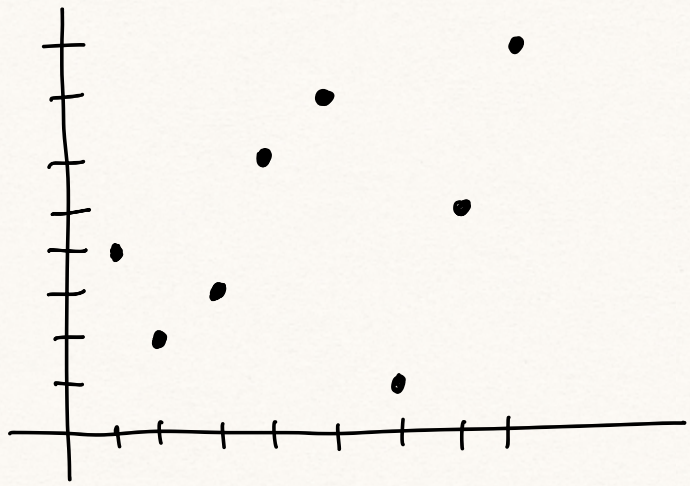
$$\begin{pmatrix} 1 & & & \\ -1 & 1 & & \\ 2 & -3 & 1 & \\ -6 & 11 & -6 & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 3 & 1 & \\ 1 & 7 & 6 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(Pf: Problems 4, 5 on HW 3 plus pages 83-84 in Stanley)

Pattern avoidance

5) Plot notation

4, 2, 3, 6, 7, 1, 5, 8



Permutation patterns (Sagan ch 1)

Def: σ contains a copy of the pattern π if some subsequence $\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_k}$ has the same relative order as $\pi_1, \pi_2, \ldots, \pi_k$.

Ex: How many 132-patterns does 42367158 have?

Def: σ avoids π if it contains no copy of π.

Def: $Av_n(\pi) = \{\pi\text{-avoiding permutations of } [n]\}$.

Q: What is $|Av_n(12)|$? $|Av_n(132)|$? $|Av_n(123)|$?