Homework 7

This homework is simply the midterm from last year. It will be scored out of 50 points and then the total score will be divided by 5 for your homework grade.

This year's midterm will aim to be similar in content and difficulty. I recommend you also study the twelvefold way and any homework problems that you did not solve.

Have fun!

Math 501: Combinatorics I, Fall 2022

Midterm Exam

INSTRUCTIONS: You have 50 minutes to complete this exam. You may not use calculators, cell phones, notes, references, or other aid besides a pen or pencil. Scratch paper will be provided. If you run out of room to write up your answer below a problem statement, continue on the back of that page.

Print your name and sign this exam on the lines below.

SCORING: Problem 1 is worth 20 total points, and problems 2 and 3 are each worth 15, with a maximum possible total score of 50. Partial credit will be given for significant progress towards a solution. The table below is for internal scoring use; do not fill it out.

Problem	Points
1	
2	
3	

Total:

HONOR PLEDGE: *This exam is my own work. I have not given, received, or used any unauthorized assistance.*

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Print Name

____

Signature

1. (Short answer - 2 points per part) Show some scratch work, since even if you get the wrong answer you might get 1 point if you had the right idea but just made a simple calculation error! (Full formal proofs not necessary for this part.)

   (a) You reach into a large bag of mixed Halloween candy with plenty of each of 4 types of candies, and take out a handful of 7 pieces of candy. How many possible combinations of candy could you have ended up with?

   (b) How many ways can you pick 3 *different* types of candies out of the bag in part (a) and give one to each of 3 specific distinct friends?

   (c) How many set partitions of $\{1, 2, 3, 4, 5\}$ have three blocks?

(d) How many binary sequences (sequences of 0s and 1s) of length 5 are there?

(e) How many binary sequences of length 10 have exactly 5 zeros and 5 ones?

(f) How many binary sequences of length 10, with 5 zeros and 5 ones, have the ballot property (that as you read from left to right, the number of 1's you have read never exceeds the number of 0's)?

(g) How many binary sequences of length 7 have no two consecutive 1s?

(h) How many labeled trees on 5 vertices are there?

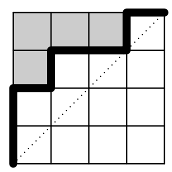
(i) How many unlabeled trees on 5 vertices are there? Draw them.

(j) For each of the trees T you drew in the previous question, how many labeled trees have unlabeled tree T? Check that your answers add up to your answer to question (e).

2. (15 points) The **Carlitz-Riordan q-Catalan numbers** $C_n(q)$ are defined as follows. Define $\mathcal{D}(n)$ to be the set of all Dyck paths of height n. Then define

$$C_n(q) = \sum_{D \in \mathcal{D}(n)} q^{\text{area}(D)}$$

where the *area* of a Dyck path D is the number of unit squares in the $n \times n$ grid lying above the path, as shown below.



area $= 4$

Also define $\mathcal{B}(n)$ to be the set of all ballot words of length $2n$, that is, permutations of $0^n 1^n$ such that, when read from left to right, the number of 1's never outweighs the number of 0s. Prove, using a weight-preserving bijection, that

$$C_n(q) = \sum_{b \in \mathcal{B}(n)} q^{\text{inv}(b)}$$

where inv is the usual inversion statistic counting the number of pairs consisting of a 1 to the left of a 0.

Write your proofs below; an extra blank page is attached before the next problem to give you plenty of room; feel free to use the backs of pages too.

3. (15 points) Let a_n be the number of sequences of numbers of length n in which:

- Each number is one of 0,1,2.
- No two 2's are consecutive.

For instance, 021100120 is a valid such sequence but 022210012 is not.

(a) (5 pts) Find a recursion for a_n (this should be similar to the Fibonacci recurrence). Remember to include the initial conditions!

(b) (5 pts) Use the recursion to find a closed form for the generating function of a_n.

(c) (5 pts) Use the formula discussed in class for solving linear recurrences to find an explicit formula for a_n in terms of n.

Write your proofs below; an extra blank page is attached to give you plenty of room; feel free to use the backs of pages too.