Math 501: Combinatorics
Homework 14

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 10. The maximum possible score on this homework is 10 points. See the syllabus for scoring details.

Problems

1. (1) [1 point] Apply Franklin's involution to the strict partition $(7, 6, 4, 3)$, and check that applying it again returns to the original partition.
2. (1+) [2 points] Sagan chapter 3 problem 17.
3. (1+) [2 points] Sagan chapter 3 problem 18.
4. (2) [3 points] Stanley chapter 1 problem 69.
5. (2) [3 points] Show that every polynomial in two variables x, y can be written in a unique way as a sum of a *symmetric polynomial* and an *antisymmetric polynomial*. Here symmetric means $f(x, y) = f(y, x)$, and antisymmetric means $f(x, y) = -f(y, x)$.
6. (2+) [4 points] Let $\mathrm{DO}(n)$ be the number of partitions into distinct odd parts. Show that

   $$\lim_{n\to\infty} \mathrm{DO}(n)/p(n) = 0.$$

   You may use the asymptotic formula for $p(n)$ discussed in class, but you may not use any known asymptotic formula online or in any textbook for $\mathrm{DO}(n)$. Instead, use combinatorics and estimation to bound $\mathrm{DO}(n)$ in terms of ordinary partition counting.

   (This fact is the main step in proving that there are, in the limit as $n \to \infty$, half as many conjugacy classes in the alternating group A_n as in the symmetric group S_n. You do not need to prove this; it's just an interesting corollary for fans of group theory!)
7. (2+) [4 points] Give a "combinatorial proof" of the rather trivial identity

   $$\prod_{i=1}^{\infty} \frac{1}{1-x^i} \cdot \prod_{i=1}^{\infty} (1-x^i) = 1$$

   by interpreting the two products on the left hand side as generating functions for counting certain partitions (possibly with sign) and give a sign-reversing involution proof of the resulting identity.
8. (3+) [10 points] Stanley chapter 1 problem 103(a).