Math 501: Combinatorics
Homework 12

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 10. The maximum possible score on this homework is 10 points. See the syllabus for scoring details.

Problems

1. (1-) [1 point] Among a family of 20 people, 9 of them like chocolate ice cream, 7 of them like vanilla ice cream, and 2 like both chocolate and vanilla. How many don't like either flavor of ice cream? Explain how the Inclusion-Exclusion principle applies here.
2. Let P be a poset in which every interval $[x, y]$ is finite. Show that, in the incidence algebra $I(P)$:
   - (1+) [2 points] f is invertible if and only if $f(x, x) \neq 0$ for all x,
   - (2-) [3 points] $fg = \delta$ if and only if $gf = \delta$ (inverses are two-sided),
   - (1) [1 point] If f is invertible then its inverse f^{-1} is unique.
3. (1+) [2 points] Sagan chapter 5 problem 17(a).
4. (2) [3 points] Sagan chapter 5 problem 37. For part (b), you only need to give one proof, not two, and you can use whatever method you like, even if it is not one of Sagan's suggested methods.
5. (2-) [3 points] Draw the Hasse diagrams of all 15 lattices on six elements. Which are modular? Distributive? Atomic?
6. (2) [3 points] Show that the number of bijective linear extensions of the poset $[2] \times [n]$ to a total ordering on $[2n]$ is given by the Catalan number C_n.
7. (1+) [2 points] Let $\sigma(n) = \sum_{d|n} d$ be the sum of divisors function. Show that $n = \sum_{d|n} \sigma(d)\mu(n/d)$ in two ways: first using Möbius inversion, and then by using a direct argument by cancelling terms on the right hand side.
8. (1+) [2 points] Describe a recursion for the square of the Möbius function, μ^2, in the incidence algebra of a poset, by thinking of it as the inverse function of ζ^2.
9. (2+) [4 points] Show that a finite graded lattice L is modular (as defined by the rank condition given in class) if and only if for all $x, y, z \in L$ with $x \leq z$, we have $x \vee (y \wedge z) = (x \vee y) \wedge z$.
10. (5) [∞ points] Stanley chapter 3 problem 135(b).