Math 501: Combinatorics
Homework 11

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 10. The maximum possible score on this homework is 10 points. See the syllabus for scoring details.

Problems

1. (1) [1 point] Show that if a least upper bound of s and t exists in a poset P, then it is unique.
2. (1) [1 point] Show that the *join-irreducible* elements (those that cannot be written as a join of smaller elements) of Young's lattice (the poset of partitions under diagram inclusion) are precisely the rectangular partitions.
3. (1) [1 point] Sagan chapter 5 problem 18(a).
4. (1+) [2 points] Draw all of the Hasse diagrams for the posets having four elements (up to isomorphism, so, unlabeled). How many are there?
5. (2) [3 points] Draw all 63 possible Hasse diagrams for the posets having five unlabeled elements. In other words, classify the posets on five-element sets up to isomorphism.
6. (2+) [4 points] The *Bruhat order* is a partial ordering on permutations defined as follows. Defining $s_i = (i\ i+1)$ to be the i-th adjacent transposition, every permutation can be written as a *reduced word* in the s_i's, that is, a product of s_i's of minimal length. Sometimes there is more than one reduced word for a permutation - for instance, the permutation (13) in S_3 can either be written as $s_1s_2s_1$ or as $s_2s_1s_2$.

   We say $\pi \leq \sigma$ in the Bruhat order if there is a reduced word for π that occurs as a subword of a reduced word for σ (not necessarily consecutively). For instance, $s_2s_3s_1$ occurs as a subword of $s_2s_4s_3s_1s_4$.

   Draw the Bruhat order for S_3, and prove that the rank generating function of the Bruhat order on S_n is $(n)_q!$.
7. (1+) [2 points] Let P and Q be graded posets. Define the product poset $P \times Q$ to be the poset on the set $P \times Q$ such that $(p, q) \leq (p', q')$ if and only if $p \leq p'$ in P and $q \leq q'$ in Q. Show that the Boolean lattice B_n is isomorphic to the product of posets $B_{n-1} \times [2]$ where $[2]$ is the poset with two elements $1, 2$ with $1 < 2$.
8. For posets P and Q, define a new poset Q^P as the set of all poset morphisms (order-preserving maps) $f : P \to Q$, where the partial ordering is given by $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in P$. Prove that, for any posets R, P, Q, we have:
   (a) (1+) [2 points] R^{P+Q} is isomorphic to $R^P \times R^Q$
   (b) (1+) [2 points] $(R^Q)^P$ is isomorphic to $R^{Q\times P}$.
9. (2+) [4 points] Stanley chapter 3 problem 12. (Hint: the minimal elements of any poset form an antichain, as do the elements that cover any given element.)
10. (3-) [8 points] Stanley chapter 3 problem 8.
11. (2+) [4 points] Stanley chapter 3 problem 23.