Math 501: Combinatorics
Homework 10

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 10. The maximum possible score on this homework is 10 points. See the syllabus for scoring details.

Problems

1. (2+) [4 points] Show that the Ramsey number $R(3, 3, 3)$ is greater than 16, by constructing an edge coloring of K_{16} in three colors that has no monochromatic triangles.

2. (1+) [2 points] Let P_n be the **path graph** with n vertices $1, 2, \ldots, n$, in which i is connected by an edge to $i + 1$ for each $i = 1, 2, \ldots, n - 1$ (and there are no other edges). Find the largest possible size of a matching for P_n, and find the smallest possible size of a maximal matching for P_n. Express your answers in terms of n (they may depend on the parity of n or its residue modulo 3).

3. (2-) [3 points] Prove that a graph is bipartite (admits a proper 2-coloring of its vertices) if and only if every cycle has even length.

4. (1) [1 point] Let m and n be positive integers with $m < n$. How many saturated matchings does the complete bipartite graph $K_{m,n}$ have?

5. (2) [3 points] (Determinant practice!) For an $n \times n$ matrix A, take the definition of $\det(A)$ to be
$$\sum_{\pi \in S_n} \operatorname{sgn}(\pi) \prod_j a_{j\pi_j}.$$
Starting with this definition, show that for any two $n \times n$ matrices A and B, we have
$$\det(AB) = \det(A)\det(B).$$

6. (3) [9 points] Prove the directed Matrix-Tree theorem directly, using the combinatorial definition for the determinant given in the previous problem.

7. (1+) [2 points] Show that Hall's Marriage Lemma is equivalent to the following statement. Let X be a finite set, and let $S_1, \ldots, S_k$ be a collection of subsets of X. Define a **transversal** of the sets S_i to be a choice of an element $x_i \in S_i$ for each i such that all of the choices x_i are distinct. Then a transversal exists if and only if, for every subset $J \subseteq [k]$,
$$\left| \bigcup_{i \in J} S_i \right| \geq |J|.$$

8. (2) [3 points] An undirected graph is ***k*-regular** if every vertex has degree k. Show that a bipartite k-regular graph must have the same number of vertices of each color in a two-coloring, and show that such a graph has a perfect matching (that saturates both vertex colors).

9. (5) [∞ points] Compute the Ramsey number $R(4, 6)$.