Math 501: Combinatorics
Final Exam Review/optional homework

This can be handed in as an optional homework; if not handed in, your score will be the average of all your homework scores so far. Only the problems marked (1+) can be handed in for a grade - there are five of them, so you must do exactly these five for a perfect score on this homework.

There are also practice problems for the final exam below. Note that there will be at least one problem on symmetric functions on the exam, so even if you do not hand in this homework, it is recommended to study problems 1-5 below.

Symmetric functions problems

1. (1+) Write the symmetric polynomial $x_1^3 + x_2^3 + x_3^3 - x_1x_2 - x_1x_3 - x_2x_3$ in $\Lambda_{\mathbb{Q}}(x_1, x_2, x_3)$ in terms of the m basis, the e basis, the h basis, the p basis, and the s basis.

2. (1+) What is the coefficient of $x_1^3x_2^2x_3x_4$ in the Schur function $s_{(4,2,1)}$? Draw the semistandard Young tableaux that you counted in order to find the coefficient.

3. (1+) Define $H(t) = \sum_{n=0}^{\infty} h_n t^n$ to be the generating function of the homogeneous symmetric functions h_n in the set of variables $X = \{x_1, x_2, \ldots\}$. Prove the identity
$$H(t) = \prod_{n=1}^{\infty} \frac{1}{1 - x_n t}.$$

4. (1+) Define $E(t) = \sum_{n=0}^{\infty} e_n t^n$ to be the generating function of the homogeneous symmetric functions e_n in the set of variables $X = \{x_1, x_2, \ldots\}$. Prove the identity
$$E(t) = \prod_{n=1}^{\infty} (1 + x_n t).$$

5. (1+) Observe that the previous two problems show that $H(t)E(-t) = 1$. Use this identity to derive a relationship between the h_n's and e_n's.

6. (5) The *chromatic symmetric function* of a (undirected) graph G on vertex set $V = \{v_1, \ldots, v_n\}$ can be defined as follows. A *proper coloring* is a map $c : V \to \mathbb{N}_{\geq 0}$ such that no two adjacent vertices in the graph are assigned the same label by c. The *monomial* associated to c is
$$x_c := x_{c(v_1)} x_{c(v_2)} \cdots x_{c(v_n)}.$$

   Finally, the chromatic symmetric function X_G is defined to be $\sum_c x_c$ where the sum is over all proper colorings c of G.

   Many of these chromatic symmetric functions have nice positivity properties in terms of common symmetric function bases. One such class of them is given by "incomparability graphs". Given a poset P, the incomparability graph is the graph on the elements of the poset in which an edge is drawn between x and y if they are incomparable, that is, $x \not\leq y$ and $y \not\leq x$.

   A poset is said to be $(3+1)$-free if it contains no induced subposet isomorphic to the direct sum of a 1-chain and a 3-chain. Show that if G is the incomparability graph of a $(3+1)$-free poset, then X_G is e-positive, that is, it can be written as a sum of elementary symmetric functions with positive integer coefficients.

   This is known as the *Stanley-Stembridge conjecture*.

Other practice problems

1. Let P_n be the set of all permutations of $1, 2, 3, 4, \ldots, 2n$ such that, in list notation, 1 and 2 are adjacent to each other, 3 and 4 are adjacent to each other, and so on ($2i-1$ and $2i$ are adjacent for all i). For instance, 562134 is a permutation in P_3.

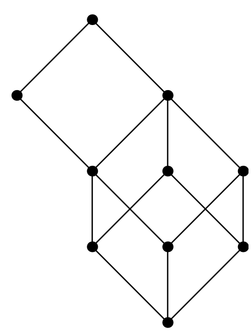
   Prove that
$$\sum_{w \in P_n} q^{\mathrm{inv}(w)} = (1+q)^n (n)_{q^4}!$$
   where the notation $(n)_{q^4}!$ denotes the q-factorial of n with q^4 plugged in for q:
$$(n)_{q^4}! = 1(1+q^4)(1+q^4+q^8)(1+q^4+q^8+q^{12})\cdots(1+q^4+q^8+\cdots+q^{4(n-1)}).$$

2. An undirected graph is **k-regular** if every vertex has degree k. Use Hall's Marriage Lemma to show that a bipartite k-regular graph admits a perfect matching.

3. Compute the number of trees on $2n$ vertices labeled by $[2n]$ in which every pair of adjacent vertices has opposite parity (that is, no two even numbers are connected and no two odd numbers are connected). (Hint: On what graph can you apply the Matrix-Tree theorem?)

4. Use the Gessel-Viennot lemma to count the number of nonintersecting pairs of Dyck paths (integer up-right lattice paths staying weakly above the diagonal line $y = x$) such that the first Dyck path goes from $(0, 0)$ to $(5, 5)$ and the second goes from $(1, 1)$ to $(4, 4)$.

5. Show that the following poset is a distributive lattice using the Fundamental Theorem of Finite Distributive Lattices. In other words, find a poset P for which the poset $J(P)$ of order ideals of P under inclusion is isomorphic to the lattice below, and show that your answer is correct.



6. Compute the Möbius number of the lattice above.

7. Prove that the exponential generating function for the number of permutations of n with exactly two cycles is
$$\frac{1}{2}\left(\ln(1-x)\right)^2$$

8. Among a group of 20 people, 14 of them like chocolate ice cream, 11 of them like vanilla ice cream, 11 like strawberry, 7 like both chocolate and vanilla, 8 like both chocolate and strawberry, and 6 like both vanilla and strawberry. Everyone likes at least one of the three flavors. How many like all three? Use the Inclusion-Exclusion principle.

9. Give a generating functions proof that the number of partitions of n into odd parts is equal to the number of partitions of n into distinct parts.