Math 501: Combinatorics
Homework 8

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 10. The maximum possible score on this homework is 10 points. See the syllabus for scoring details.

Problems

1. (1) [1 point] Sagan chapter 4 problem 11.

2. A *plane tree* is a tree along with, for every vertex v, a cyclic ordering of the nodes w connected to v by an edge. Use species to find a formula for the number of plane trees on n vertices, using the following steps.

   (a) (1+) [2 points] Build your intuition by directly counting the plane trees on n vertices for $n = 0, 1, 2, 3, 4, 5$. Write out the corresponding first handful of terms for the exponential generating function.

   (b) (1+) [2 points] Let $\mathcal{PL}$ denote the species of plane trees. Let $\mathcal{R}$ be the species of rooted plane trees, where we choose one of the n vertices to be the root. Relate $\mathcal{R}$ and $\mathcal{PL}$ with a species identity.

   (c) (2-) [3 points] Let $\mathcal{T}$ be the species of rooted trees with a *linear* order (permutation) assigned to the children of each vertex starting with the root (rather than a cyclic order). Prove that $\mathcal{R} = X \cdot (\mathcal{C} \circ \mathcal{T})$ where $\mathcal{C}$ is the species of cyclic orderings, and that $\mathcal{T} = X \cdot (\mathcal{L} \circ \mathcal{T})$ where $\mathcal{L}$ is the species of permutations in list notation.

   (d) (2+) [4 points] Solve for the exponential generating functions of $\mathcal{T}$ and $\mathcal{R}$ using the formulas you found above. Then expand the coefficients of $\mathcal{R}$ by differentiating both sides of your formula and using the generalized binomial theorem. Finally, deduce the coefficients of the egf of $\mathcal{PL}$ from your formula for those of $\mathcal{R}$.

3. (2) [3 points] (Stanley chapter 5 problem 5.27) Find a formula for the number $e(n)$ of trees with $n+1$ unlabeled vertices and n labeled *edges*. Give a simple bijective proof.

4. (2-) [3 points] Let $\mathcal{PF}$ be the species of parking functions (drawn as labeled Dyck paths with the obvious relabeling rule). Let $\mathcal{SPF}$ be the species of **strict** parking functions, that is, parking functions whose Dyck path stays **strictly** above the diagonal (except for the two endpoints). Prove that $\mathcal{PF} - 1 = (X \cdot \mathcal{SPF})'$ and that $\mathcal{PF} - 1 = \mathcal{SPF} \cdot \mathcal{PF}$.

5. (2+) [4 points] Let $\mathcal{PF}$ be the species of parking functions defined in the previous problem. Use a direct combinatorial argument to show, on the level of species, that

$$\mathcal{PF} = E \circ (X \cdot \mathcal{PF}).$$

   (Do NOT use the formula $(n+1)^{n-1}$ we already derived for the number of parking functions and comparisons with species of trees.)

6. (2-) [3 points] Solve Stanley chapter 5 problem 1(a) using exponential generating functions and/or species. The problem states: Each of n (distinguishable) telephone poles is painted red, white, blue, or yellow. An odd number are painted blue and an even number are painted yellow. In how many ways can this be done?

7. (2) [3 points] Solve Stanley chapter 5 problem 1(b) using exponential generating functions and/or species. The problem states: Suppose now the colors orange and purple are also used. The number of orange poles plus the number of purple poles is even. Now how many ways are there?

8. (2+) [4 points] An *ordered rooted tree* is a rooted tree along with a specified left-to-right ordering of the children of each node starting with the root. In other words, it can be defined recursively as a root vertex along with a list of ordered rooted trees on the remaining vertices, ordered from left to right. Let $\mathcal{T}$ be the species of ordered rooted trees. Find an equation satisfied by $\mathcal{T}$ and use it to solve for its generating function. Deduce an explicit formula for the number of ordered rooted trees on n vertices. (Be very careful at this last step! Check your answer for small values of n before you hand it in.)

9. (2+) [4 points] Prove combinatorially that the chain rule for differentiation holds on species.

10. (2-) [3 points] Recall that the ordinary generating function for the Catalan numbers, $C(x)$, satisfies $xC(x)^2 = C(x) - 1$. Use Lagrange inversion to show directly from this equation that $C_n = \frac{1}{n+1}\binom{2n}{n}$, without solving for the generating function.