Math 501: Combinatorics
Homework 7

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 10. The maximum possible score on this homework is 10 points. See the syllabus for scoring details.

For maximal midterm studying, do all of the problems 1-5 (but still only hand in a subset that follows the usual scoring rules).

Problems

1. (2) [3 points] (This problem was from the Fall 2019 qual exam for 501.) Fill in the partially completed Twelvefold Way table below, using the ten remaining formulas (a)-(j) listed below. Explain how you arrived at each answer. **For maximal midterm studying benefit, try to do this without looking at the textbook or notes and then check your answers after you're done.**

   Here, K is a k-element set, N is an n-element set, and each table entry below counts the number of functions $f : K \to N$ where the elements of K and N may either be considered distinguishable or indistinguishable, and f may have no restriction or be injective or surjective.

K	N	Any function f	Injective f	Surjective f
Dist.	Dist.			
Indist.	Dist.			
Dist.	Indist.		$\begin{cases} 0 & \text{if } n < k \\ 1 & \text{if } n \geq k \end{cases}$	
Indist.	Indist.		$\begin{cases} 0 & \text{if } n < k \\ 1 & \text{if } n \geq k \end{cases}$	

   (a) $\binom{n}{k}$
   (b) $\left(\!\binom{n}{k}\!\right)$
   (c) $\left(\!\binom{n}{k-n}\!\right)$
   (d) $S(k,n)$ (Stirling number of the second kind)
   (e) $n! \cdot S(k,n)$
   (f) $S(k,n) + S(k,n-1) + \cdots + S(k,1)$
   (g) n^k
   (h) $(n)_k$ (falling factorial)
   (i) $p(k,n)$ (partitions of k into n parts)
   (j) $p(k,n) + p(k,n-1) + \cdots + p(k,1)$

2. Short answer problems (all level (1), 1 point each):

   (a) You reach into a large bag of candy with plenty of each of 6 types of candies, and take out a handful of 11 pieces of candy. How many possible combinations of candy could you have ended up with *that contain at least one of each type of candy*?

(b) How many permutations in S_5 have exactly three cycles?

(c) How many binary sequences (sequences of 0s and 1s) of length 6 are there?

(d) How many binary sequences of length 8 have exactly 5 zeros and 3 ones?

(e) How many binary sequences of length 8, with 4 zeros and 4 ones, have the ballot property (that as you read from left to right, the number of 1's you have read never exceeds the number of 0's)?

(f) How many binary sequences of length 8 have no two consecutive 1s?

(g) How many labeled trees on 6 vertices are there?

(h) How many unlabeled trees on 6 vertices are there? Draw them.

(i) For each of the trees T you drew in the previous question, how many labeled trees have unlabeled tree T? Check that your answers add up to the correct total.

3. Let c_n be the number of sequences of numbers of length n in which:

- Each number is one of 0,1,2,3.
- No two 3's are consecutive.

For instance, 0221030132 is a valid such sequence but 033112333 is not.

(a) (1+) [2 points] Find a recursion for c_n (this should be similar to the Fibonacci recurrence). Remember to include the initial conditions!

(b) (2-) [3 points] Use the recursion to find a closed form for the generating function of c_n.

(c) (2) [3 points] Use the formula discussed in class for solving linear recurrences to find an explicit formula for c_n in terms of n.

4. (1+) [2 points] Recall that we defined the q-analog $\binom{n}{k}_q$ to be

$$\sum_{w \in S_{0^k 1^{n-k}}} q^{\mathrm{inv}(w)}.$$

Let $P_{k,n-k}$ denote the set of all up-right lattice paths from $(0,0)$ to $(k, n-k)$ in the plane, and for $p \in P_{k,n-k}$ define $\mathrm{area}(p)$ to be the number of unit squares in the $k \times (n-k)$ grid that lie above path p. Show that the q-analog

$$\sum_{p \in P_{k,n-k}} q^{\mathrm{area}(p)}$$

is also equal to $\binom{n}{k}_q$, by finding a weight-preserving bijection from $P_{k,n-k}$ to $S_{0^k 1^{n-k}}$ that sends area to inv.

5. (2) [3 points] Carlitz defined the *q-Stirling numbers of the second kind* as follows. Given a set partition B of n into k blocks, let its blocks be $B_1, B_2, \ldots, B_k$ where the blocks are written in order by their minimum element from least to greatest. Then an *inversion* of B is a pair (b, B_j) where b is in some block B_i to the left of B_j (that is, $i < j$) and $b > \min(B_j)$. For instance, in the set partition:

$$\{1,3,6\}, \{2,4\}, \{5,7\},$$

we have the inversions $(3, \{2,4\})$, $(6, \{2,4\})$, and $(6, \{5,7\})$. We write $\mathrm{inv}(B)$ for the total number of inversions, and define

$$S_q(n,k) = \sum_B q^{\mathrm{inv}(B)}$$

where B ranges over all set partitions of $[n]$ into k blocks.

Prove that the q-Stirling numbers satisfy the recursion

$$S_q(n,k) = S_q(n-1,k-1) + (1 + q + q^2 + \cdots + q^{k-1}) S_q(n-1,k).$$

6. (2+) [4 points] Compute the exponential generating function of the species of permutations having exactly two (nonempty) cycles.