Math 501: Combinatorics
Homework 3

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 10. The maximum possible score on this homework is 10 points. See the syllabus for scoring details.

See Files in Canvas for Stanley and Sagan problems.

Problems

1. (1-) [1 point] Simplify the following the composition of permutations in cycle notation. Express your answer both in cycle notation and in list notation.

$$(134)(5976) \circ (15)(279)$$

2. (1+) [2 points] Prove that every permutation can be written as a composition of transpositions (2-cycles). In other words, show that any list can be sorted by swapping numbers in pairs one step at a time. If you solve the latter version of the question, explain why it is equivalent to the former.

3. **Multinomial coefficients.** Let n be a positive integer and let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a composition of n (meaning that each λ_i is a positive integer and $\sum \lambda_i = n$). Define

$$\binom{n}{\lambda} = \binom{n}{\lambda_1, \lambda_2, \ldots, \lambda_k}$$

to be the number of distinct rearrangements of the letters in the word $1^{\lambda_1} 2^{\lambda_2} \cdots k^{\lambda_k}$ (where the exponent indicates the multiplicity of the letter - for instance, the notation $1^3 2^4 3^2$ refers to the word 111222233.) For instance, the number of ways to rearrange MISSISSIPPI is $\binom{11}{4,4,2,1}$.

(a) (1) [1 point] Show that

$$\binom{n}{\lambda} = \binom{n}{\lambda_1}\binom{n-\lambda_1}{\lambda_2}\cdots\binom{n-\lambda_1-\cdots-\lambda_{k-1}}{\lambda_k} = \frac{n!}{\lambda_1!\cdots\lambda_k!}.$$

Verify that $\binom{n}{k} = \binom{n}{k,n-k}$.

(b) (1+) [2 points] Give a combinatorial proof that

$$\binom{n}{\lambda} = \sum_i \binom{n-1}{\lambda^{(i)}}$$

where $\lambda^{(i)}$ is the partition of $n-1$ formed by reducing the i-th part by 1 (and then re-ordering the parts from greatest to least). For instance, if $\lambda = (3,2,2,1)$, then $\lambda^{(1)} = (2,2,2,1)$, $\lambda^{(2)} = (3,2,1,1)$, $\lambda^{(3)} = (3,2,1,1)$, and $\lambda^{(4)} = (3,2,2)$.

(c) (1+) [2 points] Show that these are indeed "multinomial coefficients" in the sense that $\binom{n}{\lambda}$ is the coefficient of $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_k^{\lambda_k}$ in

$$(x_1 + x_2 + \cdots + x_k)^n.$$

4. (1+) [2 points] Prove that

$$x^n = \sum_k S(n,k)(x)_k$$

where here $(x)_k = (x)(x-1)\cdots(x-k+1)$. (Hint: You may treat x as a variable and prove it using induction and the Stirling number recursion. Another approach is to treat x as an integer as in the binomial theorem problem on the last homework.)

5. (2) [3 points] Prove that

$$(x)_n := (x)(x-1)(x-2)\cdots(x-n+1) = \sum_k s(n,k)x^k,$$

where $s(n,k) = (-1)^{n-k}c(n,k)$ is the Stirling number of the first kind. If you use the recursion for $c(n,k)$, prove the recursion before you use it.

6. (2) [3 points] Show that, if a permutation π can be written as a reduced word using an even number of transpositions, then $\mathrm{inv}(\pi)$ is even. Similarly, if it can be written as a product of an odd number of transpositions, then $\mathrm{inv}(\pi)$ is odd. Conclude that the number of transpositions in any two reduced words for a permutation must have the same parity (even or oddness).

   Thus we can define the *sign* of a permutation π, denoted $\mathrm{sgn}(\pi)$, to be 1 if it is a product of an even number of transpositions and -1 if it is a product of an odd number of transpositions. Show that

$$\mathrm{sgn}(\pi \circ \sigma) = \mathrm{sgn}(\pi)\mathrm{sgn}(\sigma)$$

   for any two permutations π and σ.

7. (2+) [4 points] Sagan chapter 1 problem 36 (this may be of interest to those of you interested in computer science).

8. (2) [3 points] Stanley chapter 1 problem 44 (a)

9. (3-) [8 points] Stanley chapter 1 problem 44 (b)