Math 501: Combinatorics
Homework 2

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 10. The maximum possible score on this homework is 10 points. See the syllabus for scoring details.

NOTE: All problems from Stanley are from the second edition, which is available on Canvas under Files.

Problems

1. (1+) [2 points] Prove that addition and multiplication of cardinalities satisfies the distributive law, that is, that $|A| \cdot (|B| + |C|) = |A| \cdot |B| + |A| \cdot |C|$, using the definition of addition and multiplication of cardinalities that we defined in class.

2. (2+) [4 points] Prove the binomial theorem using a combinatorial argument as follows. Show that, for all *positive integers* s, t, and n, we have

   $$(s+t)^n = \sum_{k=0}^{n} \binom{n}{k} s^k t^{n-k}$$

   In particular, do not treat s and t as variables; rather, interpret $(s+t)^n$ as counting something parameterized by the integers s, t, n and show that the right hand side counts the same thing.

   Then, defining the polynomials $p(x) = (x+1)^n$ and $q(x) = \sum_{k=0}^{n} \binom{n}{k} x^k$, we have that $p(s) = q(s)$ for all positive integers s. Use the fact that polynomials in one variable that agree on infinitely many values must be the same to conclude that $p(x) = q(x)$ as polynomials. Finally, plug in x/y and clear the denominators on both sides of the equation $p(x/y) = q(x/y)$ to show that the binomial theorem holds:

   $$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

3. (1) [1 point] Stanley chapter 1 problem 3(e).

4. (2-) [3 points] Stanley chapter 1 problem 3(f).

5. (2) [3 points] Stanley chapter 1 problem 13.

6. (2-) [3 points] Stanley chapter 1 problem 17 (b).

7. (2-) [3 points] Stanley chapter 1 problem 43. (Hint: See Sagan chapter 1 problems 13, 14 as well for extra hints/answers. But full proofs are still needed.)

8. (3-) [8 points] Stanley chapter 1 problem 46 (b).