Math 501: Combinatorics
Homework 14

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 10. The maximum possible score on this homework is 10 points. See the syllabus for scoring details.

Problems

1. (1-) [1 point] BONUS: What is your favorite bijection, and why? (This problem does not count towards your 10 point total problem limit; it will be an extra fun point added on top to whatever you hand in for real below.)
2. (1) [1 point] Apply Franklin's involution to the partition $(7, 6, 4, 3)$, and check that applying it again returns to the original partition.
3. (1+) [2 points] Sagan chapter 3 problem 16(c).
4. (2-) [3 points] Sagan chapter 3 problem 16(d).
5. (1+) [2 points] Sagan chapter 3 problem 17.
6. (1+) [2 points] Sagan chapter 3 problem 18.
7. (2) [3 points] Stanley chapter 1 problem 69.
8. (2-) [3 points] Give a generating functions proof that the number of partitions of n into odd parts is equal to the number of partitions of n into distinct parts.
9. (2+) [4 points] Give a bijective combinatorial proof that the number of partitions of n into odd parts is equal to the number of partitions of n into distinct parts.
10. (2+) [4 points] Let $\mathrm{DO}(n)$ be the number of partitions into distinct odd parts. Show that

$$\lim_{n\to\infty} \mathrm{DO}(n)/p(n) = 0.$$

   You may use the asymptotic formula for $p(n)$ discussed in class, but you may not use any known asymptotic formula online or in any textbook for $\mathrm{DO}(n)$. Instead, use combinatorics and estimation to bound $\mathrm{DO}(n)$ in terms of ordinary partition counting.

   (This fact is the main step in proving that there are, in the limit as $n \to \infty$, half as many conjugacy classes in the alternating group A_n as in the symmetric group S_n. You do not need to prove this; it's just an interesting corollary for fans of group theory!)
11. (2+) [4 points] Give a "combinatorial proof" of the rather trivial identity

$$\prod_{i=1}^{\infty} \frac{1}{1-x^i} \cdot \prod_{i=1}^{\infty} (1-x^i) = 1$$

   by interpreting the two products on the left hand side as generating functions for counting certain partitions (possibly with sign) and give a sign-reversing involution proof of the resulting identity.
12. (3+) [10 points] Stanley chapter 1 problem 103(a).
13. (4) [10 points] Define $Q(n)$ to be the number of partitions of n into distinct parts. It is known that

$$Q(5n+1) \equiv 0 \pmod 4.$$

Show that Dyson's rank (the difference of the width and height of a partition) gives a combinatorial interpretation of this fact, that is, that the number of partitions of $5n+1$ into distinct parts whose rank is congruent to i mod 4 is equal to the number of partitions of $5n+4$ whose rank is congruent to j mod 4, for any $i, j \in \{0, 1, 2, 3\}$.

(This is a result due to Monks, i.e. the maiden name version of the professor.)

14. (5) [∞ points] Find a combinatorial proof (via a direct bijection) that the number of partitions of $5n+4$ whose rank is congruent to i mod 5 is equal to the number of partitions of $5n+4$ whose rank is congruent to j mod 5, for any $i, j \in \{0, 1, 2, 3, 4\}$.