# Math 501: Combinatorics
# Homework 11

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 10. The maximum possible score on this homework is 10 points. See the syllabus for scoring details.

## Problems

1. (1) [1 point] Sagan chapter 5 problem 17(a).
2. (1) [1 point] Sagan chapter 5 problem 17(b).
3. (1) [1 point] Sagan chapter 5 problem 18(a).
4. (1) [1 point] Sagan chapter 5 problem 18(b).
5. (1+) [2 points] Describe a natural graded poset structure on $S_n$ whose Hasse diagram is connected and whose rank generating function in the variable $q$ is $(n)_q!$.
6. (1+) [2 points] Let $P$ and $Q$ be graded posets with rank generating functions $F_P(x)$ and $F_Q(x)$. Define the product poset $P \times Q$ to be the poset on the set $P \times Q$ such that $(p, q) \leq (p', q')$ if and only if $p \leq p'$ in $P$ and $q \leq q'$ in $Q$. Show that $P \times Q$ is a graded poset with rank generating function
$$F_{P\times Q}(x) = F_P(x) \cdot F_Q(x).$$
7. For posets $P$ and $Q$, define a new poset $Q^P$ as the set of all poset morphisms (order-preserving maps) $f : P \to Q$, where the partial ordering is given by $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in P$. Prove that, for any posets $R, P, Q$, we have:
   (a) (1+) [2 points] $R^{P+Q}$ is isomorphic to $R^P \times R^Q$
   (b) (1+) [2 points] $(R^Q)^P$ is isomorphic to $R^{Q\times P}$.
8. (2) [3 points] Draw all 63 possible Hasse diagrams for the posets having five unlabeled elements. In other words, classify the posets on five-element sets up to isomorphism.
9. (2+) [4 points] Stanley chapter 3 problem 12. (Hint: the minimal elements of any poset form an antichain, as do the elements that cover any given element.)
10. (2+) [4 points] Show that a finite graded lattice $L$ is modular (as defined by the rank condition given in class) if and only if for all $x, y, z \in L$ with $x \leq z$, we have $x \vee (y \wedge z) = (x \vee y) \wedge z$.
11. (3-) [8 points] Stanley chapter 3 problem 8.
12. (2+) [4 points] Stanley chapter 3 problem 23.