Math 501: Combinatorics
Homework 10

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 10. The maximum possible score on this homework is 10 points. See the syllabus for scoring details.

Problems

1. (1+) [2 points] Let P_n be the **path graph** with n vertices $1, 2, \dots, n$, in which i is connected by an edge to $i+1$ for each $i = 1, 2, \dots, n-1$ (and there are no other edges). Find the largest possible size of a matching for P_n, and find the smallest possible size of a maximal matching for P_n. Express your answers in terms of n (they may depend on the parity of n or its residue modulo 3).

2. (2-) [3 points] Prove that a graph is bipartite if and only if every cycle has even length.

3. (2+) [4 points] The *complete bipartite graph* $K_{m,n}$ is the undirected graph on $[m+n]$ in which there is an edge between i and j if and only if $i \in \{1, 2, \dots, m\}$ and $j \in \{m+1, \dots, n\}$ (or vice versa). Show that the number of spanning trees of $K_{m,n}$ is $m^{n-1}n^{m-1}$.

4. (1) [1 point] Let m and n be positive integers with $m < n$. How many saturated matchings does the complete bipartite graph $K_{m,n}$ have?

5. (2+) [4 points] For an $n \times n$ matrix A, take the definition of $\det(A)$ to be

$$\sum_{\pi \in S_n} \operatorname{sgn}(\pi) \prod_j a_{j\pi_j}.$$

   Starting with this definition, show that for any two $n \times n$ matrices A and B, we have

$$\det(AB) = \det(A)\det(B).$$

6. (3) [9 points] Prove the directed Matrix-Tree theorem directly, using the combinatorial definition for the determinant given in the previous problem.

7. (2+) [4 points] Find, with proof, the Ramsey number $R(3,4)$.

8. (1+) [2 points] Show that Hall's Marriage Lemma is equivalent to the following statement. Let X be a finite set, and let $S_1, \dots, S_k$ be a collection of subsets of X. Define a **transversal** of the sets S_i to be a choice of an element $x_i \in S_i$ for each i such that all of the choices x_i are distinct. Then a transversal exists if and only if, for every subset $J \subseteq [k]$,

$$\left| \bigcup_{i \in J} S_i \right| \geq |J|.$$

9. (2) [3 points] An undirected graph is ***k*-regular** if every vertex has degree k. Show that a bipartite k-regular graph must have the same number of vertices of each color in a two-coloring, and show that such a graph has a perfect matching (that saturates both vertex colors).

10. (5) [∞ points] Compute the Ramsey number $R(4,6)$.