Math 501: Combinatorics, Fall 2019

Final Exam (take-home portion)

INSTRUCTIONS: The work on this exam must be entirely your own. You may use any references you wish, but you may not discuss the problems with any other individual or group of people, either online or in person, outside of office hours.

If you choose to handwrite your problems, print this document and use the space provided below each problem, plus any additional attached white sheets of paper that you need behind each problem. Staple the entire document and **include this signed cover page.**

If you choose to type your solutions, be sure to clearly label the problems, and type them up in the correct order (corresponding to the numbering on this exam). **Print your solutions and this cover page and staple it to the front of your work.** Make sure you sign the bottom of the document.

Hand in your completed exam in room E104 at 4:00 PM on Tuesday, Dec 17.

DUE DATE: Dec 17, 2019, 4:00 PM

SCORING: Each of the ten problems on this exam is worth ten points. Some may have multiple parts. For each problem or part, partial credit will be given sparingly for incomplete solutions that have made significant progress. The table below is for internal scoring use; do not fill it out.

Problem	Points	Problem	Points
1		6	
2		7	
3		8	
4		9	
5		10	

Total:

AGREEMENT TO WORK INDEPENDENTLY: *I certify that all of the attached work is my own, and I did not discuss these problems outside of office hours with any individual or group, in person or online.*

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Signature

1. **True/False:** (1 point each) Determine whether each of these is true or false. Write out the full word "True" or "False" after each question rather than just the letters "T" or "F".

   (a) There are exactly 20 paths from $(0,0)$ to $(3,3)$ using only upwards and rightwards unit steps.

   (b) If you have $s(100,50)$ pigeons and you stuff them into $S(100,50)$ holes, then some hole must contain at least two pigeons. (Here $s(n,k)$ and $S(n,k)$ are the Stirling numbers of the first and second kind, respectively.)

   (c) A generating function $F(x) = f_0 + f_1 x + f_2 x^2 + \cdots$ has a compositional inverse if and only if $f_0 \neq 0$.

   (d) A digraph has an Eulerian circuit if and only if it is *balanced*, that is, every vertex v has the same indegree as its outdegree.

   (e) For any nonempty poset P with finite chains and antichains, P has a *minimal* element.

   (f) For any nonempty poset P with finite chains and antichains, P has a *minimum* element.

   (g) An element f of the incidence algebra of a locally finite poset P has a multiplicative inverse if and only if $f(x,x) \neq 0$ for all $x \in P$.

   (h) A regular tetrahedron has exactly 24 symmetries, including reflections and rotations.

   (i) Among any six people, there are either three that are mutually acquainted or three that are mutually unacquainted.

   (j) Every lattice has a $\hat{0}$ and a $\hat{1}$.

2. **Short Answer:** (1 point each) You do not need to provide a proof or even scratchwork for these questions, just an answer!

   (a) The ten parts (a) through (j) of this question are listed in mixed-up order! Compute inv of this list, where we assign the number 1 to part (a), 2 to (b), and so on in alphabetical order.

   (f) Find a closed form expression for the summation

$$\sum_{k=0}^{n} k^2 \binom{n}{k}$$

   in terms of n.

   (e) Find the number of paths from $(0,0)$ to $(10,10)$ using only upwards and rightwards unit steps that stay *strictly* above the diagonal - that is, the path must only touch the diagonal line $y = x$ at the two endpoints $(0,0)$ and $(10,10)$, and otherwise the path lies above this line.

   (c) The ten parts (a) through (j) of this question are listed in mixed-up order! Apply the Foata bijection φ to this list, where we assign the number 1 to part (a), 2 to (b), and so on in alphabetical order.

   (j) Compute the number of spanning trees in the complete bipartite graph $K_{3,3}$.

   (b) The ten parts (a) through (j) of this question are listed in mixed-up order! Compute maj of this list, where we assign the number 1 to part (a), 2 to (b), and so on in alphabetical order.

(i) Compute

$$\sum_{\pi \in S_{2019}} (-1)^{\mathrm{inv}(\pi)}.$$

(d) The ten parts (a) through (j) of this question are listed in mixed-up order! Write this permutation in cycle notation, where we assign the number 1 to part (a), 2 to (b), and so on in alphabetical order.

(g) Find an infinite product formula for the generating function

$$\sum_{n=0}^{\infty} DE(2n)x^n$$

where $DE(m)$ is the number of partitions of m into distinct even parts.

(h) Draw the Cayley graph for the dihedral group D_4 of symmetries of the unit square, using the two generators given by the 90 degree clockwise rotational symmetry, and the vertical reflectional symmetry.

3. Recall that

$$\sum_{\pi \in S_n} q^{\mathrm{maj}(\pi)} = \sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)}.$$

In class we proved this by showing that the Foata bijection is a weight-preserving bijection $\varphi : (S_n, \mathrm{maj}) \to (S_n, \mathrm{inv})$. In this problem we will find another weight-preserving bijection as a composition of two other bijections, as follows:

(a) (2 points) Define

$$C_n = \{(c_0, c_1, \ldots, c_{n-1}) : c_i \leq i \text{ for all } i\}$$

Show that

$$\sum_{c \in C_n} q^{c_0 + \cdots + c_{n-1}} = \mathbf{n}! = (1)(1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1})$$

(b) (4 points) Find a weight-preserving bijection $(S_n, \mathrm{maj}) \to (C_n, \sum)$, and prove that your answer works.

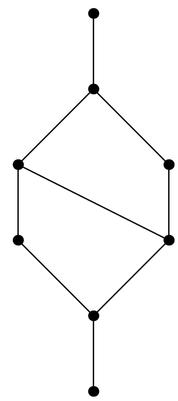
(c) (4 points) Find a weight-preserving bijection $(C_n, \sum) \to (S_n, \mathrm{inv})$, and prove that your answer works.

The composition of a pair of possible answers for parts (b) and (c) above is known as the *Carlitz* bijection.

4. Find, with proof, a simple formula for the number of spanning trees of the undirected graph K_n' formed by deleting one edge from a complete graph K_n on n labeled vertices.

5. An **ordered set partition** of $[n]$ is an ordered tuple $(A_1, A_2, \ldots, A_k)$ of nonempty sets that partition $[n]$. Use the theory of combinatorial species to find a closed formula for the exponential generating function of the sequence OSP_n, where OSP_n is the total number of ordered set partitions of $[n]$.

6. Show that the following poset is a distributive lattice using the Fundamental Theorem of Finite Distributive Lattices.



7. **Young's Lattice** is the poset $\leq_Y$ on all partitions of any size, where $\nu \leq_Y \lambda$ if and only if $\nu_i \leq \lambda_i$ for all i. Find and prove a formula for the values of the Möbius function $\mu(\nu, \lambda)$ for two partitions $\nu \leq_Y \lambda$.

8. A *tournament* T on the vertex set $[n]$ is a directed graph on $[n]$ with no loops such that each pair of distinct vertices is joined by exactly one directed edge. The weight $w(e)$ of a directed edge e from i to j (denoted $i \to j$) is defined to be x_j if $i < j$ and $-x_j$ if $i > j$. The *weight* of T is defined to be $w(T) = \prod_e w(e)$, where e ranges over all edges of T.

   Prove that
   $$\det(x_i^{j-1})_{i,j=1}^n = \sum_T w(T) = \prod_{1\le i<j\le n} (x_j - x_i)$$
   where the middle summation is over all tournaments T on $[n]$.

   (It is highly recommended that you look at the solutions in Stanley's book for problem 34 in chapter 2, in order to solve this problem. Even if you already solved some of the parts of this problem on your homework, you must write out all parts, a complete proof, in full to get full credit on this problem.)

9. Use Burnside's Lemma to count the number of inequivalent colorings of the *edges* of a cube either black or white, up to rotation.

10. Let $\mathbb{F}_q$ be the finite field of order q. Find, with proof, a simple formula for the number of surjective linear transformations $A : \mathbb{F}_q^n \to \mathbb{F}_q^k$ in terms of q, n, and k.