Lander Ver Hoef
Co-authors: SCARY $\setminus \{\text{Me}\}$
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Math 501: Homework 8

2. Let $d > 1$ be a positive integer. A *d-ary de Bruijn sequence* of degree n is a sequence of length d^n containing every length n sequence in $\{0, 1, 2, \ldots, d-1\}^n$ exactly once as a circular factor.

(a) (1+) [2 points] Show that there always exists a d-ary de Bruijn sequence of degree n for any n.

Proof. We begin by constructing the d-ary de Bruijn graph, which we will call B_d^n. Construct a digraph where the vertex set is labeled with all possible length $n-1$ sequences with characters from $\{0, 1, 2, \ldots, d-1\}$. There is an edge from vertex v_i to vertex v_j if the last $n-2$ characters in the label of v_i match the first $n-2$ characters of the label of v_j. We will label the edge from v_i to v_j by the character appended to the last $n-2$ characters of v_i to obtain the label on v_j.

We first note that for a given vertex v_i, there are precisely d vertices that have the same $n-2$ first characters as the last $n-2$ characters in the label of v_i; thus, the out-degree of v_i is d. Similarly, there are exactly d vertices with that have labels ending in the same $n-2$ characters as the first $n-2$ characters as the label on v_i, so the in-degree of v_i is also d. Thus, B_d^n is balanced.

Moreover, we claim that B_d^n is connected. Given two vertices v_i, v_j with labels $a_1 a_2 \ldots a_{n-1}$ and $b_1 b_2 \ldots b_{n-1}$ respectively, we form the concatenation $a_1 a_2 \ldots a_{n-1} b_1 b_2 \ldots b_{n-1}$. The first $n-1$ characters label the start vertex. Incrementing our search window by 1, we find that $a_2 a_3 \ldots a_{n-1} b_1$ is also a vertex in B_d^n that has an edge from v_i by our definition of B_d^n. Continuing in this manner, we find a path from v_i to v_j, and therefore B_d^n is connected. The combination of being balanced and connected shows that B_d^n contains an Eulerian tour.

We claim that an Eulerian tour of B_d^n corresponds to a d-ary de Bruijn sequence. We first observe that the edges of B_d^n are in one-to-one correspondence with the length n sequences in $\{0, 1, 2, \ldots, d-1\}^n$ by concatenating the label on an edge e on the end of the label of $\text{init}(e)$. Each $n-1$ length sequence is represented exactly once on the vertices of B_d^n, and the d possible completions of any particular sequence are represented on the d edges leaving the vertex labeled with that sequence.

Given an Eulerian tour of B_d^n, we construct a sequence b_d^n as follows: starting with the empty sequence, append the label on each edge as we reach it in the tour. Observe that because of how B_d^n was constructed, after appending a character corresponding to an edge e, the last $n-1$ characters in b_d^n (possibly wrapping around to the end of the sequence, as the starting vertex of an Eulerian tour is arbitrary) give the label of the vertex $\text{fin}(e)$, and the last n characters give the length n sequence that e is correspondence with. Therefore, because the Eulerian tour traverses each edge exactly once, the sequence b_d^n contains each length n sequence exactly once, and is therefore a de Bruijn sequence. □

(b) (2) [3 points] Find the number of d-ary de Bruijn sequences of degree n that begin with n zeroes.

Proof. By the previous part, the number of d-ary de Bruijn sequences of degree n is equal to the number of Eulerian tours in B_d^n. In particular, the d-ary de Bruijn sequences beginning with n zeroes corresponds to the Eulerian tours starting at the vertex labeled with $n-1$ zeroes where the first edge in the tour is the edge labeled 0. So, we will count the number of Eulerian tours in B_d^n that begin with a particular edge.

Let A be the adjacency matrix of B_d^n, the directed d-ary de Bruijn graph. Because B_d^n has d^{n-1} vertices (one for each distinct word of length $n-1$ with d letters), A is a $d^{n-1} \times d^{n-1}$ matrix. We observe that the proof of Lemma 5.6.13 does not rely on the fact that $d = 2$, so the holds in the case of general d as well.

Therefore, by Lemma 5.6.13, A^{n-1} is the matrix of all 1s (since the i, j entry of A^{n-1} counts the number of paths from v_i to v_j, and by Lemma 5.6.13, there is exactly one such path). As this is a rank 1 $d^{n-1} \times d^{n-1}$ matrix, it has only one non-zero eigenvalue, and by the eigenvector $(1, 1, \ldots, 1)$, that eigenvalue is d^{n-1}.

However, if A has eigenvalues $\lambda_1, \ldots, \lambda_{d^{n-1}}$, then the eigenvalues of A^{n-1} are $\lambda_1^{n-1}, \ldots, \lambda_{d^{n-1}}^{n-1}$. By taking an nth root, we find that the eigenvalues of A are $d\zeta$ once (where ζ is some $n-1$st root of unity) and 0 $d^{n-1} - 1$ times.

To determine ζ, we will use the trace of A. That is, how many vertices have loops? In B_d^n, a vertex has a loop if and only if the last $n-2$ characters in the label are the same as the first $n-2$ characters; this happens exactly when the label is $n-1$ copies of the same character. Thus, there are precisely d vertices in B_d^n with loops, so the trace of A is d. Because the sum of the eigenvalues is equal to the trace, we have that $d\zeta = d$, so $\zeta = 1$.

We now note that we can construct $L(B_d^n)$ as

$$L(B_d^n) = dI - A.$$

Because $L(B_d^n)$ is defined as a polynomial in A, we can calculate its eigenvalues by plugging each eigenvalue into that same polynomial; this gives us that the eigenvalues of $L(B_d^n)$ are d ($d^{n-1} - 1$ times, once for each 0 eigenvalue of A) and 0 once (for the single d eigenvalue of A).

By Corollary 5.6.7, we can compute the number of Eulerian tours starting at a particular edge e as

$$\epsilon(B_d^n, e) = \frac{1}{d^{n-1}} d^{d^{n-1}-1} \left((d-1)!\right)^{d^{n-1}} = d^{d^{n-1}-n} ((d-1)!)^{d^{n-1}} = \frac{1}{d^n} (d!)^{d^{n-1}}.$$

□

3. (2−) [3 points] An *undirected Eulerian tour* is a tour on the edges of an undirected graph using every undirected edge exactly once (in just one direction). Derive necessary and sufficient conditions for the existence of an undirected Eulerian tour in an undirected graph. Prove your result.

Solution: The necessary and sufficient conditions for the existence of an Eulerian tour are that the graph must be connected with every vertex having an even degree. It is clear that if an Eulerian tour exists, these conditions hold; connectedness is trivial, and the even degree is a consequence of the fact that the tour must enter and leave each vertex via paired edges without reusing any edge, so each vertex must have even degree.

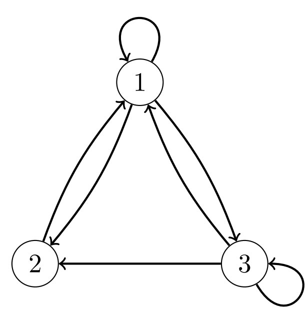
To see that these conditions are sufficient, suppose G is a connected graph with every vertex having even degree. We first show that G contains a tour: starting at some vertex v_1, begin by following some edge e_1 to another vertex v_2. Then, because v_2 has even degree and the tour has used an odd number of edges at v_2 (in this case, only e_1), there must be at least one unused edge e_2 that we can leave along. Continuing in this fashion, we observe that at each step, if we arrive at a vertex v_j other than v_1, the tour must have used an odd number of edges incident to v_j, as any possible previous visits used edges in pairs (entering and leaving), and the tour just used a single edge to arrive at v_j. Thus, there must be at least one edge remaining to leave along. The only vertex at which it is possible for this algorithm to terminate (i.e., for there to be no unused edges to follow) is the starting vertex v_1; when the tour arrives back at v_1 (which it must eventually do, since there are only finitely many vertices and we can't "get stuck" anywhere else) it will have used an even number of edges at v_1, as the arriving edge and the very first edge used e_1 will balance each other out. Thus, there is a tour in G.

Let $\mathcal{E}$ be a tour in G of maximal length, and suppose to produce a contradiction that $\mathcal{E}$ is not Eulerian. That is, there exists some edge e in G that is not used. Without loss of generality, suppose that e is incident to a vertex on $\mathcal{E}$ (if it were not, there would be another unused edge that was incident, because G is connected). Then consider $G' = G \setminus \mathcal{E}$. Because G is balanced and $\mathcal{E}$ is a tour, G' is also balanced. Consider the connected component of G' that contains e. This component, H', is connected and balanced, so by the previous paragraph, H' contains a tour starting at e. We can thus extend $\mathcal{E}$ by following $\mathcal{E}$ up to the point where it meets the tour in H' for the first time, following that tour, then continuing along $\mathcal{E}$. This contradicts our assumption that $\mathcal{E}$ is of maximal length, so E is Eulerian.

4. The *adjacency matrix* of a directed graph $D = (V, E)$ on a vertex set $V = \{1, 2, \dots, n\}$ is the matrix $A = (a_{ij})$ whose i, j entry is 1 if $(i, j) \in E$ and 0 otherwise.

   (d) (2−) [3 points] Let b_n be the number of sequences of length $n + 1$ with entries from $\{1, 2, 3\}$ that start with 1, end with 3, and do not contain the subsequences 22 or 23. Find a closed formula for the generating function of b_n using the transfer-matrix method, by constructing a directed graph in which certain paths are counted by b_n. You may use a computer to calculate the determinants, but you must write out the directed graph and the corresponding matrices.

   Proof. Let D be the digraph shown below:



   Then b_n also counts the number of paths in D starting at 1 and ending at 3. Note that D has adjacency matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

   To use the transfer matrix method we will need to compute the determinant of

$$I - Ax = \begin{pmatrix} 1-x & -x & -x \\ -x & 1 & 0 \\ -x & -x & 1-x \end{pmatrix}.$$

   This has determinant

$$(1-x)^2 - x^3 - x^2 - x^2(1-x) = 1 - 2x - 3x^2.$$

We will also need to take the determinant $\det(I - Ax; 3, 1)$, where this is found by taking determinant of the minor obtained by removing the 3rd row and 1st column; that is, we are computing

$$\begin{vmatrix} -x & -x \\ 1 & 0 \end{vmatrix} = x.$$

Thus, by the transfer-matrix method, a closed form for the generating function for b_n is given by

$$\frac{(-1)^{1+3}x}{1 - 2x - 3x^2} = \frac{x}{1 - 2x - 3x^2}.$$

$\square$