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Math 501: Homework 3

1. Let n be a positive integer and let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition of n (recall that this means that each λ_i is a positive integer and that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$). Define

$$\binom{n}{\lambda} = \binom{n}{\lambda_1, \lambda_2, \ldots, \lambda_k}$$

to be the number of distinct rearrangements of letters in the word $1^{\lambda_1} 2^{\lambda_2} \cdots k^{\lambda_k}$ (where the exponent indicates the multiplicity of the letter – for instance, the notation $1^3 2^4 3^2$ refers to the word 111222233). This is classically known as the 'MISSISSIPPI' problem, since it gives a formula for the number of ways of rearranging the letters in the word MISSISSIPPI. The symbol $\binom{n}{\lambda}$ is often referred to as a "multinomial coefficient."

(a) (1) [1 point] Show that

$$\binom{n}{\lambda} = \binom{n}{\lambda_1}\binom{n-\lambda_1}{\lambda_2} \cdots \binom{n-\lambda_1-\cdots-\lambda_{k-1}}{\lambda_k} = \frac{n!}{\lambda_1! \cdots \lambda_k!}.$$

Verify that $\binom{n}{k} = \binom{n}{k, n-k}$.

Proof. We begin counting the number of rearrangmenets of $1^{\lambda_1} 2^{\lambda_2} \cdots k^{\lambda_k}$ by counting the number of ways of choosing which of the n places in the word are filled by the λ_1 1's, which is $\binom{n}{\lambda_1}$. We then choose which of the $n - \lambda_1$ spots are filled by the λ_2 2's, which is $\binom{n-\lambda_1}{\lambda_2}$, and so forth for each λ_i. Thus,

$$\binom{n}{\lambda} = \binom{n}{\lambda_1}\binom{n-\lambda_1}{\lambda_2} \cdots \binom{n-\lambda_1-\cdots-\lambda_{k-1}}{.}$$

If we use the formula that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, we find that

$$\binom{n}{\lambda_1}\binom{n-\lambda_1}{\lambda_2} \cdots \binom{n-\lambda_1-\cdots-\lambda_{k-1}}{\lambda_k}$$
$$= \frac{n!}{\lambda_1!(n-\lambda_1)!} \frac{(n-\lambda_1)!}{(\lambda_2)!(n-\lambda_1-\lambda_2)!} \cdots \frac{(n-\lambda_1-\cdots-\lambda_{k-1})}{(\lambda_k)!(n-\lambda_1-\cdots-\lambda_k)!},$$

which, after canceling terms and recognizing that $(n - \lambda_1 - \cdots - \lambda_k)! = 0! = 1$, gives the desired expression

$$\binom{n}{\lambda} = \frac{n!}{\lambda_1 \cdots \lambda_k!}.$$

□

(b) (1+) [2 points] Give a combinatorial proof that

$$\binom{n}{\lambda} = \sum_i \binom{n-1}{\lambda^{(i)}}$$

where $\lambda^{(i)}$ is the partition of $n - 1$ formed by reducing the i-th part by 1 (and then re-ordering the parts from greatest to least). For instance, if $\lambda = (3, 2, 2, 1)$, then $\lambda^{(1)} = (2, 2, 2, 1), \lambda^{(2)} = (3, 2, 1, 1), \lambda^{(3)} = (3, 2, 1, 1,)$, and $\lambda^{(4)} = (3, 2, 2)$.

Proof. We observe that $\binom{n-1}{\lambda^{(i)}}$ counts the number of rearrangements of $1^{\lambda_1}2^{\lambda_2}\cdots k^{\lambda_k}$ that have the character i in the first position. So, we can find the total number of rearrangements of $1^{\lambda_1}2^{\lambda_2}\cdots k^{\lambda_k}$ by summing over i. □

(c) (1+) [2 points] Show that these are indeed "multinomial coefficients" in the sense that $\binom{n}{\lambda}$ is the coefficient of $x_1^{\lambda_1}x_2^{\lambda_2}\cdots x_k^{\lambda_k}$ in

$$(x_1 + x_2 + \cdots + x_k)^n.$$

Proof. We consider $(x_1 + x_2 + \cdots + x_k)^n$ as

$$(x_1 + x_2 + \cdots + x_k)(x_1 + x_2 + \cdots + x_k)\cdots(x_1 + x_2 + \cdots + x_k), \tag{1}$$

then consider the coefficient on $x_1^{\lambda_1}x_2^{\lambda_2}\cdots x_k^{\lambda_k}$ as the number of ways of assembling this product. Given a rearrangement of the word $W = 1^{\lambda_1}2^{\lambda_2}\cdots k^{\lambda_k}$, we can associate positions in W with terms in the product in Equation 1. Then if W has an i in position j, we choose x_i to be the element of term n that appears in our monomial. This is a bijection between rearrangements of W and ways that the monomial $x_1^{\lambda_1}x_2^{\lambda_2}\cdots x_k^{\lambda_k}$ can appear in the expansion of Equation 1, so the coefficient is $\binom{n}{\lambda}$. □

2. (1+) [2 points] Use the recursion for the Stirling numbers of the second kind to give a proof by induction on n that

$$x^n = \sum_k S(n,k)(x)_k$$

(where here x is just a variable, not a positive integer.)

Proof. First, observe that $x^1 = S(1,1)(x)_1$, which simplifies to the tautology $x = x$, showing that the base case holds.

Next, observe that

$$x^{n+1} = xx^n = x\sum_{k=1}^{n} S(n,k)(x)_k = \sum_{k=1}^{n} S(n,k)x(x)_k \tag{2}$$

by the inductive hypothesis. Recalling the recursion for $(x)_k$, that

$$(x)_k = (x)_{k-1}(x - k + 1),$$

we reindex k to see that

$$(x)_{k+1} = (x)_k(x - k) = x(x)_k - k(x)_k.$$

Solving this for $x(x)_k$ gives us that

$$x(x)_k = (x)_{k+1} + k(x)_k.$$

Substituting this into Equation 2 we obtain

$$\sum_{k=1}^{n} S(n,k)((x)_{k+1} + k(x)_k) = \sum_{k=1}^{n} S(n,k)(x)_{k+1} + \sum_{k=1}^{n} kS(n,k)(x)_k.$$

We reindex the first sum to see that this is equivalent to

$$\sum_{k=2}^{n+1} S(n,k-1)(x)_k + \sum_{k=1}^{n} kS(n,k)(x)_k = \sum_{k=1}^{n+1}(S(n,k-1) + kS(n,k))(x)_k, \tag{3}$$

where the combining of sums with different summation bounds is justified by the fact that $S(n,k) = 0$ for all $k < 1$ and $k > n$. We now recall the recursion for $S(n,k)$:

$$S(n,k) = S(n-1,k-1) + kS(n-1,k).$$

Reindexing this in n, we see that

$$S(n+1,k) = S(n,k-1) + kS(n,k).$$

If we substitute this recursion into the right-hand side of Equation 3, we obtain

$$\sum_{k=1}^{n+1} S(n+1,k)(x)_k,$$

which is precisely what we needed to show. □

4. (2) [3 points] Show that, if a permutation π can be written as a reduced word using an even number of transpositions, then $\operatorname{inv}(\pi)$ is even. Similarly, if it can be written as a product of an odd number of transpositions, then $\operatorname{inv}(\pi)$ is odd. Conclude that the number of transpositions in any two reduced words for a permutation must have the same parity (even or oddness).

   Thus, we can define the *sign* of a permutation π, denoted $\operatorname{sgn}(\pi)$, to be 1 if it is a product of an even number of transpositions and -1 if it is a product of an odd number of transpositions. Show that

   $$\operatorname{sgn}(\pi \circ \sigma) = \operatorname{sgn}(\pi)\operatorname{sgn}(\sigma)$$

   for any two permutations π and σ.

Proof. We can view the permutation π in list form as being obtained by applying π as a product of adjacent transpositions S_i to the simply ordered word $1\,2\,3\ldots n$. Observe that $\operatorname{inv}(1\,2\,3\ldots n) = 0$, and that applying S_i to any permutation in list form changes inv by precisely 1: if i and $i+1$ were properly ordered before, then after the application of S_i, they are in reverse order, and inv has increased by one, while if they were in reverse order before, then after they are properly ordered and inv has decreased by 1.

Therefore, if π can be written as a reduced word using an even number of transpositions, $\operatorname{inv}(\pi)$ differs from 0 by an even amount, and as 0 is even, $\operatorname{inv}(\pi)$ must be even. Similarly, $\operatorname{inv}(\pi)$ differs from 0 by an odd amount if π can be written as a reduced word using an odd number of transpositions. Since $\operatorname{inv}(\pi)$ depends only on π, not on the particular form of the reduced word representing π, the parity of the number of transpositions in any reduced word for π must be invariant.

To see that $\operatorname{sgn}(\pi \circ \sigma) = \operatorname{sgn}(\pi)\operatorname{sgn}(\sigma)$, we observe that to get $\pi \circ \sigma$ into reduced word format, we are only cancelling identical adjacent transpositions that appear consecutively in $\pi \circ \sigma$; that is, if $S_i S_i$ appears in $\pi \circ \sigma$, we remove them. Note that in doing so, we are always removing precisely two transpositions, so the parity of the number of transpositions in $\pi \circ \sigma$ is unchanged. Therefore, $\operatorname{sgn}(\pi \circ \sigma)$ is odd if and only if exactly one of $\operatorname{sgn}(\pi)$ or $\operatorname{sgn}(\sigma)$ are odd, which is precisely the condition needed to make $\operatorname{sgn}(\pi \circ \sigma) = \operatorname{sgn}(\pi)\operatorname{sgn}(\sigma)$ hold. □