Math 501: Combinatorics
Homework 9

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 10. The maximum possible score on this homework is 10 points. See the syllabus for scoring details.

Problems

1. (1+) [2 points] Prove that every tree is bipartite.
2. (1) [1 point] Prove that every cycle in a bipartite graph has even length.
3. (2-) [3 points] Prove that a graph is bipartite if and only if every cycle has even length.
4. (2+) [4 points] For an $n \times n$ matrix A, take the definition of $\det(A)$ to be
$$\sum_{\pi \in S_n} \operatorname{sgn}(\pi) \prod_j a_{j\pi_j}.$$
Starting with this definition, show that for any two $n \times n$ matrices A and B, we have
$$\det(AB) = \det(A)\det(B).$$
5. (3) [9 points] Prove the directed Matrix-Tree theorem directly, using the combinatorial definition for the determinant given in the previous problem.
6. (2+) [4 points] Prove the (directed) "Matrix-Forest" theorem, which is the following 'q-analog' of the Matrix-Tree theorem. Given a directed graph $D = (V, E)$ on n vertices, define the *augmented Laplacian* to be $L_q(D) = L(D) + qI_n$ where q is a formal variable and I_n is the $n \times n$ identity matrix. Show that
$$\det(L_q(D)) = \sum_{k=1}^{n} F_k(D) q^k$$
where $F_k(D)$ is the number of **oriented spanning forests** of D having k trees, that is, forests that use all vertices in V such that every tree in the forest is an oriented tree towards some root. Thus, in particular, an oriented spanning forest having k trees has k specified roots.
7. (2+) [4 points] Find, with proof, the Ramsey number $R(3, 4)$.
8. (1+) [2 points] Show that Hall's Marriage Lemma is equivalent to the following statement. Let X be a finite set, and let $S_1, \ldots, S_k$ be a collection of subsets of X. Define a **transversal** of the sets S_i to be a choice of an element $x_i \in S_i$ for each i such that all of the choices x_i are distinct. Then a transversal exists if and only if, for every subset $J \subseteq [k]$,
$$\left| \bigcup_{i \in J} S_i \right| \geq |J|.$$
9. (2) [3 points] An undirected graph is k**-regular** if every vertex has degree k. Show that a bipartite k-regular graph must have the same number of vertices of each color in a two-coloring, and show that such a graph has a perfect matching (that saturates both vertex colors).
10. (5) [∞ points] Compute the Ramsey number $R(4, 6)$.