Math 501: Combinatorics
Homework 6

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 10. The maximum possible score on this homework is 10 points. See the syllabus for scoring details.

NOTE: All problems from Stanley are from the second edition, which is available for free on his website here: http://www-math.mit.edu/∼rstan/ec/ec1.pdf

Problems

1. (2-) [3 points] Solve Stanley chapter 5 problem 1(a) using exponential generating functions and/or species.

2. (2) [3 points] Solve Stanley chapter 5 problem 1(b) using exponential generating functions and/or species.

3. (2-) [3 points] Prove that $e^{xe^{xe^{x\cdots}}}$ makes sense as a formal power series. That is, show that the sequence y_k defined by $y_0 = x$ and for all $k \geq 1$, $y_k = xe^{xy_{k-1}}$ converges in the ring of formal power series.

4. (1) [1 point] Let $\mathcal{T}$ be the species of labeled trees, and let $\mathcal{F}$ be the species of labeled forests. Show that $\mathcal{F} = \mathcal{E} \circ \mathcal{T}$ where $\mathcal{E}$ is the trivial species.

5. (1+) [2 points] Let $\mathcal{C}$ be the species of nonempty cyclic permuations (in cycle notation) and let $\mathcal{P}_{\text{cyc}}$ be the species of permutations in cycle notation. Show that $\mathcal{P}_{\text{cyc}} = \mathcal{E} \circ \mathcal{C}$, and check that this makes sense on the level of exponential generating functions by computing the generating function $\mathcal{C}(x)$ directly.

6. (2+) [4 points] An *ordered rooted tree* is a rooted tree along with a specified left-to-right ordering of the children of each node starting with the root. In other words, it can be defined recursively as a root vertex along with a list of ordered rooted trees on the remaining vertices, ordered from left to right. Let $\mathcal{T}$ be the species of ordered rooted trees. Find an equation satisfied by $\mathcal{T}$ and use it to solve for its generating function. Deduce an explicit formula for the number of ordered rooted trees on n vertices. (Be very careful at this last step! Check your answer for small values of n before you hand it in.)

7. (3-) [8 points] A *plane tree* is a tree along with, for every vertex v, a cyclic ordering of the nodes w connected to v by an edge. Use species to find a formula for the number of plane trees on n vertices.

8. (2+) [4 points] The **derivative** of a species $\mathcal{F}$ is the species $\mathcal{F}'$ defined by:

   - $\mathcal{F}'(A) = \mathcal{F}(A \cup \{*\})$ where $*$ is a formal symbol representing an element not in A
   - For any bijection $\sigma : A \to B$, the relabeling rule $\mathcal{F}'(\sigma) : \mathcal{F}'(A) \to \mathcal{F}'(B)$ is given by the map $\mathcal{F}(\sigma')$ where $\sigma' : A \cup \{*\} \to B \cup \{*\}$ is the natural extension of σ sending $*$ to $*$.

   Use this definition to give a species proof that the chain rule for differentiation holds.

9. (2) [3 points] Stanley chapter 5 problem 5.7(a).

10. (2-) [3 points] Recall that the ordinary generating function for the Catalan numbers, $C(x)$, satisfies $xC(x)^2 = C(x) - 1$. Use Lagrange inversion to show directly from this equation that $C_n = \frac{1}{n+1}\binom{2n}{n}$, without solving for the generating function.