Math 501: Combinatorics
Homework 6

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 10. The maximum possible score on this homework is 10 points. See the syllabus for scoring details.

NOTE: All problems from Stanley are from the second edition, which is available for free on his website here: http://www-math.mit.edu/∼rstan/ec/ec1.pdf

Problems

1. (2+) [4 points] Use generating functions to prove the identity
$$\sum_{k=0}^{n} \binom{2k}{k}\binom{2(n-k)}{(n-k)} = 4^n.$$
(Hint: The generating function for the Catalan numbers may come in handy!)

2. (1+) [2 points] A **binary tree** of length n is a graph on n vertices constructed recursively as follows. The empty set is a binary tree of length 0. Otherwise a binary tree has a *root vertex* v, a *left subtree* T_1, and a *right subtree* T_2, each of which is also a binary tree having a root vertex. We draw the root vertex at the top and draw an edge going down and left from v and to the root vertex of T_1, and an edge going down and right from v to the root vertex of T_2, and draw each of T_1 and T_2 recursively in the same manner.

   Prove that the number of binary trees on n vertices is the nth Catalan number C_n.

3. (2-) [3 points] A **triangulation** of a convex $n+2$-gon is a collection of $n-1$ diagonals that do not intersect each other. Show that the number of triangulations of a convex $n+2$-gon is the nth Catalan number C_n.

4. (1+) (2 points) A **ballot sequence** of length $2n$ is a permutation w of the word $(1)^n(-1)^n$ such that every partial sum $w_1 + \cdots + w_k$ is nonnegative. For instance, $(1)(-1)(1)(1)(-1)(-1)$ is ballot, but $(1)(-1)(-1)(1)(1)(-1)$ is not. Show that the number of ballot sequences of length $2n$ is the nth Catalan number C_n.

5. (2) (3 points) Recall that the nth Bell number B_n is the number of partitions of an n-element set $[n]$ into nonempty blocks. On a previous homework, we proved that
$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k.$$
Use this relation to prove that the exponential generating function of the Bell numbers is
$$\sum_{n=0}^{\infty} \frac{B_n}{n!}x^n = e^{e^x-1}.$$

6. (2+) [4 points] A **derangement** of $[n]$ is a permutation $\pi \in S_n$ having no fixed points, that is, $\pi_i \neq i$ for all i. Let D_n be the number of derangements of $[n]$. Prove that
$$\sum_{n=0}^{\infty} \frac{D_n}{n!}x^n = \frac{e^{-x}}{1-x}.$$

7. From Stanley's book *Catalan Numbers*: Here is an elegant way to prove the explicit formula combinatorially.

(a) (2+) [4 points] Let X_n be the set of all $\binom{2n}{n}$ lattice paths from $(0,0)$ to (n,n) with steps $(0,1)$ and $(1,0)$. Define the *excedance* of a path $P \in X_n$ to be the number of integers i such that at least one point (i,i') of P lies above the line $y=x$ (i.e., $i' > i$). Show that the number of paths in X_n with excedance j is independent of j.

(b) (1) [1 point] Deduce that the number of $P \in X_n$ that never rise above the line $y=x$ is given by the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.

8. (2) [3 points] Stanley chapter 1 problem 101.

9. (5) [∞ points] Let P be a Dyck path of height n, that is, a path from $(0,0)$ to (n,n) using only up and right unit steps and staying weakly above the diagonal $x=y$. Define the *bounce path* of P as follows: starting at $(0,0)$, draw a path upwards until it reaches the beginning of a rightward step in P. Then turn right and continue drawing the path until it hits the diagonal $y=x$. Then turn upwards again until it hits the beginning of another rightward step, and so on. If the bounce path hits the diagonal at points $(0,0)=(j_0,j_0),(j_1,j_1),\ldots,(j_k,j_k)=(n,n)$, define the *bounce* of P to be

$$\mathrm{bounce}(P) = \sum_{i=1}^{k-1}(n-j_i).$$

Also define the *area* of P to be the number of complete unit squares between P and the line $x=y$. Then define

$$C_n(q,t) = \sum_{D\in\mathcal{P}_\setminus} q^{\mathrm{area}(D)} t^{\mathrm{bounce}(D)}$$

where $\mathcal{P}_n$ is the set of all Dyck paths of height n.

The quantity $C_n(q,t)$ is a natural q,t-analog of the Catalan numbers that arises in the study of certain doubly graded S_n-modules. Due to this algebraic connection, it is known that

$$C_n(q,t) = C_n(t,q).$$

Find a combinatorial proof of this symmetry relation.

10. (5) [∞ points] It is known that the "super Catalan numbers"

$$S(m,n) = \frac{(2m)!(2n)!}{m!n!(m+n)!}$$

have the following properties:

- $C_n = \frac{1}{2}S(1,n)$
- $\binom{2n}{n} = S(0,n)$
- $\frac{1}{2}S(m,n)$ is an integer unless $(m,n)=(0,0)$

Find a combinatorial interpretation of $\frac{1}{2}S(m,n)$.