Math 501: Combinatorics
Homework 5

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 10. The maximum possible score on this homework is 10 points. See the syllabus for scoring details.

NOTE: All problems from Stanley are from the second edition, which is available for free on his website here: http://www-math.mit.edu/~rstan/ec/ec1.pdf

Problems

1. (1-) (1 point) Attend Maria Gillespie's talk in RMAC (the Rocky Mountain Algebraic Combinatorics seminar) on Friday, September 27. Details:

   - Cookies are in Weber 117 at 3:30 PM
   - Talks start at 4 PM in Weber 223
   - Maria's talk is at 5 PM in Weber 223 (this part gives the 1 point)
   - Dinner is afterwards

   Special rules for this problem: You may do this **in addition** to any other valid set of homework problems. If the set of other homework problems you hand in totals to less than 10 points, this will boost your score by 1, but you may hand in 10 points worth and still come to this seminar. At the seminar, Maria will have a piece of paper for you to put your name on so she remembers which of you attended the talk.

   If you come to the talk, it may be helpful to try problems 2 and 3 below beforehand.

2. (2) (3 points) Let R_n denote the number of permutations $\pi \in S_n$ for which $\pi_1 = 1$, $\pi_n = n$, and $|\pi_i - \pi_{i+1}| \leq 2$ for all $i \leq n-1$. Show that $R_0 = 0$, $R_1 = 1$, $R_2 = 1$, and for all $n \geq 3$,
$$R_n = R_{n-1} + R_{n-3}.$$
   Use this recursion to find a closed form for the generating function of the sequence R_n.

3. (2) (3 points) Use generating functions to derive an explicit formula for the Fibonacci numbers F_n, defined by $F_0 = 0$, $F_1 = 1$, and for all $n \geq 2$,
$$F_n = F_{n-1} + F_{n-2}.$$

4. (1+) (2 points) Show that multiplication of formal power series is associative.

5. (1+) (2 points) Show that the product rule for differentiation holds for formal power series.

6. (2-) (3 points) Let $F_1(x), F_2(x), \ldots$ and $F(x)$ be formal power series in $\mathbb{C}[[x]]$. Show that, if $\lim_{i\to\infty} F_i(x) = F(x)$, then
$$\lim_{i\to\infty} \frac{d}{dx} F_i(x) = \frac{d}{dx} F(x).$$

7. (2+) (4 points) Show that the chain rule holds for formal power series. (Hint: Use the previous two problems!)

8. (2) (3 points) Let $p(n,k)$ be the number of partitions of n into exactly k nonzero parts. Show that
$$\sum_{n,k} p(n,k) y^k x^n = \prod_{k=0}^{\infty} \frac{1}{1 - yx^k}.$$