Math 501: Combinatorics
Homework 4

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 10. The maximum possible score on this homework is 10 points. See the syllabus for scoring details.

NOTE: All problems from Stanley are from the second edition, which is available for free on his website here: http://www-math.mit.edu/~rstan/ec/ec1.pdf

Problems

In all of the following, **boldface** math is used to denote the standard q-analog of the symbol in bold. For instance, $\mathbf{3} = 1+q+q^2$, and $\mathbf{n}! = 1\cdots(1+q)\cdot(1+q+q^2)\cdot\cdots\cdot(1+q+\cdots+q^{n-1})$.

1. (1) [1 point] Apply the Foata bijection φ to the permutation $\pi = 526731498$, and verify that $\mathrm{maj}(\pi) = \mathrm{inv}(\varphi(\pi))$.

2. (1) [1 point] Apply the inverse of Foata's bijection, φ^{-1}, to the permutation $\alpha = 526731498$, and verify that $\mathrm{inv}(\alpha) = \mathrm{maj}(\varphi^{-1}(\alpha))$.

3. Let $P_{k,m}$ be the set of all paths from $(0,k)$ to $(m,0)$ in the plane using only unit steps of $(1,0)$ and $(0,-1)$ (going right or down) at each step. Define the statistic

$$\mathrm{area} : P_{k,m} \to \mathbb{N}$$

by defining $\mathrm{area}(p)$ to be the area of the region in the first quadrant that lies below the path p.

   (a) (2-) [3 points] Show that there is a weight-preserving bijection from $(P_{k,m}, \mathrm{area})$ to $(S_{1^k 2^m}, \mathrm{inv})$. Conclude that $|P_{k,m}|_q = \binom{\mathbf{m}+\mathbf{k}}{\mathbf{k}}$.

   (b) (1+) [2 points] Let $Q_{6,6}$ be the subset of $P_{6,6}$ consisting of the paths that pass through the point $(3,3)$. Compute $|Q_{6,6}|_q$, where the statistic is the area function restricted to $Q_{6,6}$.

4. (2-) [3 points] Prove the q-binomial theorem:

$$\prod_{j=1}^{n}(1+xq^j) = \sum_{k=0}^{n} q^{k(k+1)/2}\binom{\mathbf{n}}{\mathbf{k}}x^k.$$

You may use either induction or a combinatorial argument.

5. (2+) [4 points] Consider a noncommutative ring over $\mathbb{R}$ generated by two variables x and y, having the 'noncommutative multiplication' rule

$$yx = qxy,$$

where $q \in \mathbb{R}$ is some nonzero constant. Assume that multiplication is still distributive across addition, and all constants commute with each other and all variables. For instance, $x \cdot 3$ still equals $3 \cdot x$, but $x^2 y$ is not equal to yx^2.

Prove that, in this algebra, we have the 'quantum binomial theorem'

$$(x+y)^n = \sum_{k=0}^{n} \binom{\mathbf{n}}{\mathbf{k}} x^k y^{n-k}.$$

6. The q-multinomial coefficient may be defined as:

$$\binom{\mathbf{n}}{\lambda_{\mathbf{1}}, \ldots, \lambda_{\mathbf{k}}} = \sum_{w \in S_{1^{\lambda_1} 2^{\lambda_2} \cdots k^{\lambda_k}}} q^{\mathrm{inv}(w)}$$

where λ is any partition of n. Here

$$\mathrm{inv}(w) = |\{(i,j) : i < j, w_i > w_j\}|.$$

(a) (2-) [3 points] Show that

$$\binom{\mathbf{n}}{\lambda} = \sum_{i=1}^{k} q^{n-\lambda_1-\cdots-\lambda_i} \binom{\mathbf{n-1}}{\lambda^{(\mathbf{i})}}$$

where $\lambda^{(i)}$ is the partition defined in Homework 3.

(b) (2-) [3 points] Show that

$$\binom{\mathbf{n}}{\lambda} = \frac{\mathbf{n}!}{\prod_i \lambda_{\mathbf{i}}!}.$$

(c) (2+) [4 points] Can you find and prove a q-analog of the multinomial theorem? Make sure your answer reduces to one of the two q-analogs of the binomial theorem mentioned above when $k = 2$.

7. (5) [∞ points] Let λ be a partition of n. Define $F(\lambda)$ to be the set of *fillings* of λ, that is, ways of labeling the squares of the Young diagram of λ with the numbers $1, \ldots, n$.

For a filling σ, let $c^{(1)}, \ldots, c^{(m)}$ be the columns of σ, written as words read from top to bottom. Then we define $\mathrm{maj}(\sigma) = \sum_{i=1}^{m} \mathrm{maj}(c^{(i)})$. For instance, in the filling

5	7	
2	8	3
4	1	6

the major index is

$$\mathrm{maj}(524) + \mathrm{maj}(781) + \mathrm{maj}(36) = 1 + 2 + 0 = 3.$$

A *relative inversion* of σ is a pair of entries u and v in the same row, with u to the left of v, such that if b is the entry directly below u, either $u > v$ and it is not the case that $v < b < u$, or $u < v$ and $u < b < v$. (If u and v are in the bottom row, we set $b = 0$.) The above filling has the three relative inversions $(4,1)$, $(2,8)$, and $(8,3)$. We define $\mathrm{inv}(\sigma)$ to be the number of relative inversions of σ.

It is known that

$$\sum_{\sigma \in F(\lambda)} q^{\mathrm{inv}(\sigma)} t^{\mathrm{maj}(\sigma)} = \sum_{\rho \in F(\lambda^*)} q^{\mathrm{maj}(\sigma)} t^{\mathrm{inv}(\sigma)}$$

where λ^* is the *conjugate partition* formed by transposing the Young diagram. Find a combinatorial proof of this identity.