Math 501: Combinatorics
Homework 3

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 10. The maximum possible score on this homework is 10 points. See the syllabus for scoring details.

NOTE: All problems from Stanley are from the second edition, which is available for free on his website here: http://www-math.mit.edu/~rstan/ec/ec1.pdf

Problems

1. Let n be a positive integer and let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition of n (recall that this means that each λ_i is a positive integer and that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$). Define

$$\binom{n}{\lambda} = \binom{n}{\lambda_1, \lambda_2, \ldots, \lambda_k}$$

to be the number of distinct rearrangements of the letters in the word $1^{\lambda_1} 2^{\lambda_2} \cdots k^{\lambda_k}$ (where the exponent indicates the multiplicity of the letter - for instance, the notation $1^3 2^4 3^2$ refers to the word 111222233.) This is classically known as the 'MISSISSIPPI' problem, since it gives a formula for the number of ways of rearranging the letters in the word MISSISSIPPI. The symbol $\binom{n}{\lambda}$ is often referred to as a "multinomial coefficient".

   (a) (1) [1 point] Show that

$$\binom{n}{\lambda} = \binom{n}{\lambda_1}\binom{n-\lambda_1}{\lambda_2}\cdots\binom{n-\lambda_1-\cdots-\lambda_{k-1}}{\lambda_k} = \frac{n!}{\lambda_1! \cdots \lambda_k!}.$$

   Verify that $\binom{n}{k} = \binom{n}{k, n-k}$.

   (b) (1+) [2 points] Give a combinatorial proof that

$$\binom{n}{\lambda} = \sum_i \binom{n-1}{\lambda^{(i)}}$$

   where $\lambda^{(i)}$ is the partition of $n-1$ formed by reducing the i-th part by 1 (and then re-ordering the parts from greatest to least). For instance, if $\lambda = (3,2,2,1)$, then $\lambda^{(1)} = (2,2,2,1)$, $\lambda^{(2)} = (3,2,1,1)$, $\lambda^{(3)} = (3,2,1,1)$, and $\lambda^{(4)} = (3,2,2)$.

   (c) (1+) [2 points] Show that these are indeed "multinomial coefficients" in the sense that $\binom{n}{\lambda}$ is the coefficient of $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_k^{\lambda_k}$ in

$$(x_1 + x_2 + \cdots + x_k)^n.$$

2. (1+) [2 points] Use the recursion for the Stirling numbers of the second kind to give a **proof by induction** on n that

$$x^n = \sum_k S(n,k)(x)_k$$

(where here x is just a variable, not a positive integer.)

3. (2+) [4 points] Prove (by any method) that

$$(x)_n = \sum_k s(n,k) x^k,$$

where $s(n,k) = (-1)^{n-k} c(n,k)$ is the Stirling number of the first kind. If you use the recursion for $c(n,k)$, prove the recursion before you use it.

4. (2) [3 points] Show that, if a permutation π can be written as a reduced word using an even number of transpositions, then $\mathrm{inv}(\pi)$ is even. Similarly, if it can be written as a product of an odd number of transpositions, then $\mathrm{inv}(\pi)$ is odd. Conclude that the number of transpositions in any two reduced words for a permutation must have the same parity (even or oddness).

   Thus we can define the *sign* of a permutation π, denoted $\mathrm{sgn}(\pi)$, to be 1 if it is a product of an even number of transpositions and -1 if it is a product of an odd number of transpositions. Show that

   $$\mathrm{sgn}(\pi \circ \sigma) = \mathrm{sgn}(\pi)\mathrm{sgn}(\sigma)$$

   for any two permutations π and σ.

5. (2) [3 points] Stanley chapter 1 problem 20

6. (2) [3 points] Stanley chapter 1 problem 44 (a)

7. (3-) [8 points] Stanley chapter 1 problem 44 (b)