Math 501: Combinatorics
Homework 2

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 10. The maximum possible score on this homework is 10 points. See the syllabus for scoring details.

NOTE: All problems from Stanley are from the second edition, which is available for free on his website here: http://www-math.mit.edu/~rstan/ec/ec1.pdf

2ND NOTE: Stanley has very abbreviated solutions for many of these problems in his book. In order to receive credit, you must give a full proof that is more fleshed out than the "solution" in Stanley. Finally, I highly recommend not looking at the solution hints at all. Doing them on your own will help you keep up with the rest of the course, and after this homework I will be relying much less on Stanley's problems and the homeworks will consist much more of problems of my own design.

Problems

1. (2+) [4 points] Prove the binomial theorem using a combinatorial argument as follows. Show that, for all *positive integers* s, t, and n, we have

$$(s+t)^n = \sum_{k=0}^{n} \binom{n}{k} s^k t^{n-k}$$

   In particular, do not treat s and t as variables; rather, interpret $(s+t)^n$ as counting something parameterized by the integers s, t, n and show that the right hand side counts the same thing.

   Then, defining the polynomials $p(x) = (x+1)^n$ and $q(x) = \sum_{k=0}^{n} \binom{n}{k} x^k$, we have that $p(s) = q(s)$ for all positive integers s. Use the fact that polynomials in one variable that agree on infinitely many values must be the same to conclude that $p(x) = q(x)$ as polynomials. Finally, plug in x/y and clear the denominators on both sides of the equation $p(x/y) = q(x/y)$ to show that the binomial theorem holds:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

2. (1) [1 point] Stanley chapter 1 problem 3(e).

3. (2-) [3 points] Stanley chapter 1 problem 3(f).

4. (2) [3 points] Stanley chapter 1 problem 13.

5. (2-) [3 points] Stanley chapter 1 problem 17 (b).

6. (1+) [2 points] Stanley chapter 1 problem 107.

7. (2+) [4 points] Stanley chapter 1 problem 108 (a).

8. (2-) [3 points] Suppose $f, g : [k] \to [n]$ are equivalent with $[k]$ indistinguishable. Show that f is injective if and only if g is injective, and that f is surjective if and only if g is surjective.

   Show the same things for two functions $f, g : [k] \to [n]$ which are equivalent with $[n]$ indistinguishable, or with both indistinguishable.

   This shows that the twelvefold way entries are well-defined.