Math 501: Combinatorics
Homework 14

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 10. The maximum possible score on this homework is 10 points. See the syllabus for scoring details.

Problems

1. (1) [1 point] Let a finite group G act on a finite set X, and let $x \in X$. Show that the stabilizer $\mathrm{Stab}_G(x)$ is a subgroup of G.

2. (2-) [3 points] Two elements g_1, g_2 of a group G are *conjugate* if there exists $h \in G$ such that $g_1 = h^{-1} g_2 h$. Show that:

   (a) Conjugacy is an equivalence relation,

   (b) Two permutations in S_n are conjugate if and only if they have the same cycle type (multiset of sizes of their cycles in cycle notation)

   (c) Two conjugate elements in G always have the same number of fixed points in any action of G on a set X.

3. (2-) [3 points] How many symmetries (including rotations and reflections) does a regular dodecahedron have? A regular icosahedron? Use the Orbit-Stabilizer theorem.

4. (2) [3 points] How many inequivalent ways, up to rotation, can you color each face of a cube either red, white, or green? Use Burnside's lemma.

5. (2+) [4 points] How many inequivalent ways, up to rotation, can you color the faces of a regular dodecahedron either black or white? Use Burnside's lemma.

6. (2+) [4 points] Use Burnside's lemma to prove the following identity:

$$\frac{1}{2^n} \sum_{k=0}^{n} (2n - 2k) \binom{n}{k} = n$$

7. (1+) [2 points] What does Burnside's lemma tell us about the average number of fixed points of any permutation of $[n]$?