Math 501: Combinatorics
Homework 13

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 10. The maximum possible score on this homework is 10 points. See the syllabus for scoring details.

Problems

1. (1) [1 point] Apply Franklin's involution to the partition $(7, 6, 4, 3)$, and check that applying it again returns to the original partition.

2. (1+) [2 points] Starting with Euler's discovery of the generating function relation

$$\prod_{i=1}^{\infty}(1-x^i) = \sum_{m=-\infty}^{\infty}(-1)^m x^{m(3m-1)/2},$$

derive the recursion

$$\begin{aligned} p(n) &= p(n-1) + p(n-2) - p(n-5) - p(n-7) + \cdots \\ &= \sum_{m=1}^{\infty}(-1)^{m-1}\left(p\left(n - \frac{m(3m-1)}{2}\right) + p\left(n - \frac{m(3m+1)}{2}\right)\right) \end{aligned}$$

that we discussed in class.

3. (2) [3 points] Stanley chapter 1 problem 69.

4. (2) [3 points] Stanley chapter 1 problem 91(b). (Part (a) is another formulation of the Jacobi triple product identity.)

5. (2) [3 points] Stanley chapter 1 problem 121(a).

6. (2-) [3 points] Give a generating functions proof that the number of partitions of n into odd parts is equal to the number of partitions of n into distinct parts.

7. (2+) [4 points] Give a "combinatorial proof" of the rather trivial identity

$$\prod_{i=1}^{\infty}\frac{1}{1-x^i}\cdot\prod_{i=1}^{\infty}(1-x^i) = 1$$

by interpreting the two products on the left hand side as generating functions for counting certain partitions (possibly with sign) and give a sign-reversing involution proof of the resulting identity.

8. (3+) [10 points] Stanley chapter 1 problem 103(a).

9. (4) [10 points] Define $Q(n)$ to be the number of partitions of n into distinct parts. It is known that

$$Q(5n+1) \equiv 0 \pmod 4.$$

Show that Dyson's rank gives a combinatorial interpretation of this fact, that is, that the number of partitions of $5n+1$ into distinct parts whose rank is congruent to i mod 4 is equal to the number of partitions of $5n+4$ whose rank is congruent to j mod 4, for any $i, j \in \{0, 1, 2, 3\}$.

(This is a result due to Monks, i.e. the maiden name version of the professor.)

10. (5) [∞ points] Find a combinatorial proof (via a direct bijection) that the number of partitions of $5n+4$ whose rank is congruent to i mod 5 is equal to the number of partitions of $5n+4$ whose rank is congruent to j mod 5, for any $i, j \in \{0, 1, 2, 3, 4\}$.