# Math 501: Combinatorics
# Homework 10

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 10. The maximum possible score on this homework is 10 points. See the syllabus for scoring details.

## Problems

1. (1+) [2 points] Show that the Hasse diagram of any finite poset can always be drawn in a way such that, if $a < b$, then $a$ is drawn lower than $b$ (with a lower $y$-coordinate) on the diagram. (Hint: Induction on the size of the poset may help clean up your argument.)

   Conclude that every finite poset has a linear extension to a total ordering.

2. (1+) [2 points] Describe a graded poset whose rank generating function in the variable $q$ is $(n)_q!$.

3. (2) [3 points] Draw all 63 possible Hasse diagrams for the posets having five unlabeled elements. In other words, classify the posets on five-element sets up to isomorphism.

4. (2) [3 points] Use Dilworth's theorem to prove the **Erdös-Szekeres theorem**: that in any sequence of $nm+1$ positive integers, there is either an increasing subsequence (not necessarily consecutive) of length $n+1$ or a decreasing subsequence of length $m+1$. (Hint: What poset on the numbers has the property that increasing sequences correspond to chains and decreasing to antichains?)

5. (2+) [4 points] Stanley chapter 3 problem 12. (Hint: the minimal elements of any poset form an antichain, as do the elements that cover any given element.)

6. (1+) [2 points] Let $P$ and $Q$ be graded posets with rank generating functions $F_P(x)$ and $F_Q(x)$. Show that $P \times Q$ is a graded poset with rank generating function

$$F_{P\times Q}(x) = F_P(x) \cdot F_Q(x).$$

7. For posets $P$ and $Q$, define a new poset $Q^P$ as the set of all poset morphisms (order-preserving maps) $f : P \to Q$, where the partial ordering is given by $f \le g$ if and only if $f(x) \le g(x)$ for all $x \in P$. Prove that, for any posets $R, P, Q$, we have:

   (a) (1+) [2 points] $R^{P+Q}$ is isomorphic to $R^P \times R^Q$

   (b) (1+) [2 points] $(R^Q)^P$ is isomorphic to $R^{Q\times P}$.

8. (3-) [8 points] Stanley chapter 3 problem 8.

9. (2+) [4 points] Stanley chapter 3 problem 23.