Modular forms arising from $Q(n)$ and Dyson's rank

Maria Monks
On collaborative work with Ken Ono
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Background: Partitions

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- For instance, $(5, 3, 1, 1)$ is a partition of 10, and $(5, 3, 2)$ is a partition of 10 into distinct parts.
- $p(n)$ is the number of partitions of n.
- $Q(n)$ is the number of partitions of n into distinct parts.
- Neither $p(n)$ nor $Q(n)$ is known to have an elegant closed formula.

Background: Partitions

- Ramanujan discovered the famous congruence identities

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\begin{aligned}
p(5n+4) &\equiv 0 \pmod 5 \\
p(7n+5) &\equiv 0 \pmod 7 \\
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- The generating function for $p(n)$:

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{1-q}\frac{1}{1-q^2}\frac{1}{1-q^3}\cdots$$

- Define

$$\begin{aligned} (a;q)_\infty &= (1-a)(1-aq)(1-aq^2)(1-aq^3)\cdots \\ (a;q)_n &= (1-a)(1-aq)\cdots(1-aq^{n-1}). \end{aligned}$$

Then $\sum p(n)q^n = \frac{1}{(q;q)_\infty}$.

Principle #1:

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Behind every good partition identity
lies a q-series identity
waiting to be discovered!

Background: $p(5n+4) \equiv 0 \pmod 5$

- The generating function for $p(n)$ can be used to show

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6}.$$

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can be used to show that $Q(n)$ is nearly always divisible by 4.

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- Do the most elementary proofs of these facts require the use of generating functions and q-series manipulation?

Principle #2:

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- Atkin and Swinnerton-Dyer proved that the rank taken modulo 5 sorts the partitions of $5n + 4$ into 5 equal-sized groups!
- M. showed that Dyson's rank, taken modulo 4, sorts the partitions of n having distinct parts into four equal sized groups for nearly all positive integers n.

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- Let $Q(n,r)$ denote the number of partitions of n into distinct parts having rank r, and define

$$G(z;q) = \sum_{n,r} Q(n,r) z^r q^n.$$

- One can show that

$$G(z;q) = 1 + \sum_{s=1}^{\infty} \frac{q^{s(s+1)/2}}{(1-zq)(1-zq^2)\cdots(1-zq^s)}$$

for $z, q \in \mathbb{C}$ with $|z| \leq 1$, $|q| < 1$.

Background: $G(\pm i; q)$

- The combinatorial result involving Dyson's rank modulo 4 can be used to show that

$$qG(i;q^{24}) = \sum_{k=0}^{\infty} i^k q^{(6k+1)^2} + \sum_{k=1}^{\infty} i^{k-1} q^{(6k-1)^2}$$

and

$$qG(-i;q^{24}) = \sum_{k=0}^{\infty} (-i)^k q^{(6k+1)^2} + \sum_{k=1}^{\infty} (-i)^{k-1} q^{(6k-1)^2}.$$

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- Note that the coefficients are roots of unity and are 0 whenever the exponent of q is not a perfect square. Such functions are known as *false theta functions*.
- Resemble Ramanujan's mock theta functions, which have recently been linked to the theory of automorphic forms.

Principle #3:

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Background: Modular Forms

- Let Γ be a subgroup of $SL_2(\mathbb{Z})$. A *modular form of weight* $k \in \frac{1}{2}\mathbb{Z}$ with respect to Γ is a meromorphic function $f : \mathbb{H} \to \mathbb{C}$ such that for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$,

$$f\left(\frac{az+b}{cz+d}\right) = \epsilon(a,b,c,d)(cz+d)^k f(z),$$

where $|\epsilon(a,b,c,d)| = 1$.

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  where $|\epsilon(a,b,c,d)| = 1$.

- If we define $q = e^{2\pi i\tau}$, then

$$\eta(\tau) := q^{1/24}(q;q)_\infty$$

  is a modular form of weight $1/2$. Thus $q(q;q)_\infty^{24}$ is the Fourier expansion of a modular form of weight 12.

Background: Maass forms

- Let Γ be a subgroup of $\Gamma_0(4)$. A *harmonic weak Maass form* of weight k is a continuous modular form of weight $2-k$ with multiplier system

$$\epsilon(a,b,c,d) = \chi(d)\left(\frac{c}{d}\right)^{2k}\epsilon_d^{-2k},$$

where χ is a Dirichlet character of order 4 and $\epsilon_d = \begin{cases} 1 & d \equiv 1 \pmod 4 \\ i & d \equiv 3 \pmod 4 \end{cases}$, that is annihilated by the weight-k hyperbolic Laplacian operator

$$\Delta_k = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

and has at most linear exponential growth at the cusps of Γ.

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- Example: Let $P(n,r)$ denote the number of partitions of n having rank r, and define $R(z;q) = \sum_{n,r} P(n,r) z^r q^n$. Then

$$R(z;q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{\prod_{k=1}^{n}(1 - zq^k)(1 - z^{-1}q^k)}.$$

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- Bringmann, Ono: $qR(i; q^{24})$ is the holomorphic part of a harmonic weak Maass form.
- $R(i; q^{-1}) = R(-i; q^{-1}) = \frac{1-i}{2} G(i; q) + \frac{1+i}{2} G(-i; q)$. Thus the behaviour of $G(\pm i, q)$ gives the behavior of $R(\pm i, q)$ outside the unit disk! This also relates $G(\pm i, q)$ to automorphic forms.

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- Define the series

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Theorem (M., Ono)

We have that

$$q^{\frac{1}{12}} \cdot D(\zeta; q)D(\zeta^{-1}; q) = 4 \cdot \frac{\eta(2\tau)^4}{\eta(\tau)^2\eta(\zeta^2; 2\tau)}$$

is a weight 1 *modular form for roots of unity* $\zeta \neq \pm 1$.

$G(\omega; 1/q)$ for roots of unity ω

- We have seen that a linear combination of $G(\pm i; 1/q)$ is equal to $R(i; q)$. What happens outside the unit disk for other roots of unity ω?

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- Define $\widehat{G}(\omega; q) = G(\omega; 1/q)$. Formal manipulation yields

$$\widehat{G}(\omega; q) = \sum_{n \geq 0} \frac{(-\omega^{-1})^n}{(\omega^{-1} q; q)_n}.$$

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- This is not a well-defined q-series, but we can consider the partial sums $\widehat{G}_t(\omega; q) = \sum_{n=0}^{t} \frac{(-\omega^{-1})^n}{(\omega^{-1} q; q)_n}$.

The “difference of limits” theorem

- If $-\omega$ is a primitive mth root of unity, then the sequence formed by taking every mth term of the sequence of partial sums $\widehat{G}_1(\omega; q), \widehat{G}_2(\omega; q), \ldots$ converges to a well-defined q-series!

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Theorem (M., Ono)

Suppose that $-\omega \neq 1$ is an mth primitive root of unity. If $1 \leq r \leq m$, then $\lim_{n\to\infty} \widehat{G}_{mn+r}(\omega; q)$ is a well defined q-series and satisfies

$$\lim_{n\to\infty} \widehat{G}_{mn+r}(\omega; q) = \lim_{n\to\infty} \widehat{G}_{mn}(\omega; q) + \frac{(-\omega^{-1})^r - 1}{\omega + 1} \frac{1}{(\omega^{-1}q; q)_\infty}.$$

Example: The case $-\omega = -1$

$$
\begin{aligned}
\widehat{G}_1(1;q) &= -q - q^2 - q^3 - q^4 - q^5 - q^6 - q^7 - q^8 - \cdots \\
\widehat{G}_3(1;q) &= -q - q^2 - 2q^3 - 2q^4 - 3q^5 - 4q^6 - 5q^7 - 6q^8 - \cdots \\
\widehat{G}_5(1;q) &= -q - q^2 - 2q^3 - 2q^4 - 4q^5 - 5q^6 - 7q^7 - 9q^8 - \cdots \\
\widehat{G}_7(1;q) &= -q - q^2 - 2q^3 - 2q^4 - 4q^5 - 5q^6 - 8q^7 - 10q^8 - \cdots \\
\widehat{G}_9(1;q) &= -q - q^2 - 2q^3 - 2q^4 - 4q^5 - 5q^6 - 8q^7 - 10q^8 - \cdots
\end{aligned}
$$

and

$$
\begin{aligned}
\widehat{G}_2(1;q) &= 1 + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + 3q^7 + 4q^8 + \cdots \\
\widehat{G}_4(1;q) &= 1 + q^2 + q^3 + 3q^4 + 3q^5 + 5q^6 + 6q^7 + 9q^8 + \cdots \\
\widehat{G}_6(1;q) &= 1 + q^2 + q^3 + 3q^4 + 3q^5 + 6q^6 + 7q^7 + 11q^8 + \cdots \\
\widehat{G}_8(1;q) &= 1 + q^2 + q^3 + 3q^4 + 3q^5 + 6q^6 + 7q^7 + 12q^8 + \cdots
\end{aligned}
$$

Example: The case $-\omega = e^{2\pi i/3}$

Let $\omega = -e^{-2\pi i/3}$, and let $\zeta = e^{2\pi i/6} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$. Then

$$\begin{aligned}
\widehat{G}_1(\omega;q) &= 1 \\
\widehat{G}_4(\omega;q) &= 1 + \zeta^4 q^2 + \zeta^4 q^3 - 2q^4 + (\zeta^2 - 1)q^5 + 2\zeta^2 q^6 + \cdots \\
\widehat{G}_7(\omega;q) &= 1 + \zeta^4 q^2 + \zeta^4 q^3 - 2q^4 - 2q^5 + (\zeta^4 - 1)q^6 + \cdots
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$$\begin{aligned}
\widehat{G}_2(\omega;q) &= \zeta + \zeta q + q^2 + \zeta^5 q^3 + \zeta^4 q^4 - q^5 + \zeta^2 q^6 + \cdots \\
\widehat{G}_5(\omega;q) &= \zeta + \zeta q + q^2 + (1 + \zeta^5)q^3 + \zeta^5 q^4 - \sqrt{3}iq^5 - \sqrt{3}iq^6 + \cdots \\
\widehat{G}_8(\omega;q) &= \zeta + \zeta q + q^2 + (1 + \zeta^5)q^3 + \zeta^5 q^4 - \sqrt{3}iq^5 + \cdots
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$$\begin{aligned}
\widehat{G}_3(\omega;q) &= \zeta^2 q + \zeta^2 q^2 + \zeta q^3 + \zeta q^4 + q^5 + q^6 + \cdots \\
\widehat{G}_6(\omega;q) &= \zeta^2 q + \zeta^2 q^2 + \zeta q^3 + \sqrt{3}iq^4 + \zeta q^5 + (2 + \sqrt{3}i)q^6 + \cdots
\end{aligned}$$

Explicit formula for the case $-\omega = -1$

Theorem (M., Ono)

If we define the sequence $b(n)$ such that $\sum_{n=0}^{\infty}(-1)^n b(n) q^n = \prod_{k=0}^{\infty}(1+q^{2k+1})$, then

$$\lim_{n\to\infty} \widehat{G}_{2n}(1;q) = \frac{1}{2}\left(\sum_{n=0}^{\infty} b(n)q^n + \frac{1}{(q;q)_\infty}\right)$$

and

$$\lim_{n\to\infty} \widehat{G}_{2n+1}(1;q) = \frac{1}{2}\left(\sum_{n=0}^{\infty} b(n)q^n - \frac{1}{(q;q)_\infty}\right).$$

- The proof invokes Principle #2: Behind every good q-series identity lies a combinatorial insight waiting to be discovered!

Relating $\widehat{G}(\omega, q)$ to automorphic forms

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- Consider a twist of the third-order mock theta function of Ramanujan:

$$\psi(\omega; q) := \sum_{n\geq 0} \frac{q^{n^2}\omega^n}{(q; q^2)_n}.$$

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$$\psi(\omega; q) := \sum_{n\geq 0} \frac{q^{n^2}\omega^n}{(q; q^2)_n}.$$

- Also define

$$\widehat{D}(\omega; q) = (1+\omega^{-1})\widehat{G}(\omega; q) + (1-\omega^{-2})(\psi(-\omega^2; q) - 1).$$

Relating $\widehat{G}(\omega, q)$ to automorphic forms

Theorem (Folsom)

Let $-\omega \neq 1$ be a primitive mth root of unity. Then $q^{-1/12}\widehat{D}(\omega; q)\widehat{D}(\omega^{-1}; q)$ is the weight 1 modular form

$$q^{-1/12}\widehat{D}(\omega; q)\widehat{D}(\omega^{-1}; q) = \frac{\eta^4(q^2)\eta^2(\omega^2, q)}{\eta^2(q)\eta^3(\omega^2, q^2)}.$$

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- Thus G and $\widehat{G}$ appear within the theory of automorphic forms!

Recap

- We have started with a combinatorial observation about Dyson's rank for partitions into distinct parts, studied the relevant q-series, related these to the theory of automorphic forms, and related a kind of analytic continuation of the q-series outside the unit disk to the theory of automorphic forms.

Recap

- We have started with a combinatorial observation about Dyson's rank for partitions into distinct parts, studied the relevant q-series, related these to the theory of automorphic forms, and related a kind of analytic continuation of the q-series outside the unit disk to the theory of automorphic forms.
- We have also found a formula for $\widehat{G}(1;q)$ in terms of well-known q-series using combinatorial methods.

Recap

- We have started with a combinatorial observation about Dyson's rank for partitions into distinct parts, studied the relevant q-series, related these to the theory of automorphic forms, and related a kind of analytic continuation of the q-series outside the unit disk to the theory of automorphic forms.
- We have also found a formula for $\widehat{G}(1; q)$ in terms of well-known q-series using combinatorial methods.
- Challenges for the future:
  - We have only computed $\widehat{G}(\omega; q)$ in the case $-\omega^{-1} = -1$. What about other roots of unity?
  - Can more combinatorial results be obtained from the analytic properties of $G(\omega; q)$ or $\widehat{G}(\omega; q)$ at other roots of unity?

Sketch of Proof

- By the difference of limits theorem,

$$\lim_{n\to\infty} \widehat{G}_{2n+1}(1;q) - \lim_{n\to\infty} \widehat{G}_{2n}(1;q) = \frac{-1}{(q;q)_\infty}.$$

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- We now wish to find the sum of the limits:

$$S(q) := \lim_{n\to\infty} \widehat{G}_{2n+1}(1;q) + \lim_{n\to\infty} \widehat{G}_{2n}(1;q).$$

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- Want to show: $S(q) = \sum_{n=0}^{\infty} b(n)q^n$, where $(-1)^n b(n)$ counts the number of partitions of n into distinct odd parts.

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- Let $c(t)$ be the coefficient of q^t in $S(q)$.
- Let $\text{Even}(t)$ denote the number of partitions of t having an even number of parts.
- Let $\text{Odd}(t)$ denote the number of partitions of t having an odd number of parts.
- Using the formula for $S(q)$, one can show combinatorially that $c(t) = \text{Even}(t) - \text{Odd}(t)$.

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- It now suffices to show that $(-1)^t(\text{Even}(t) - \text{Odd}(t))$ is equal to the number of partitions of t into distinct odd parts. We show this in the case that t is even.

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  - Suppose m occurs at most once, and let $2^j m$ be the smallest even part of this form.
    - If $2^j m$ occurs an odd number of times, split one copy of $2^j m$ into two copies of $2^{j-1} m$.

Sketch of Proof

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  - If λ has distinct odd parts, do nothing.
  - Otherwise, let m be the smallest odd number such that the sum of the parts of λ of the form $2^k m$ is greater than m.
  - Suppose m occurs more than once in λ. If $2m$ occurs an even number of times, merge two parts of size m, and otherwise, split a part of size $2m$ into two parts of size m.
  - Suppose m occurs at most once, and let $2^j m$ be the smallest even part of this form.
    - If $2^j m$ occurs an odd number of times, split one copy of $2^j m$ into two copies of $2^{j-1} m$.
    - If instead $2^j m$ occurs an even number of times, merge two of them if $2^{j+1} m$ occurs an even number of times, and otherwise split one copy of $2^{j+1} m$.
- Can show that φ is an involution, and maps the partitions of n into an even number of parts and not into distinct odd parts bijectively to those having an odd number of parts.

Example: $n = 6$

$$\begin{array}{rcl} (5,1) & \circlearrowleft & \\ (4,2) & \leftrightarrow & (4,1,1) \\ (3,3) & \leftrightarrow & (6) \\ (3,1,1,1) & \leftrightarrow & (3,2,1) \\ (2,2,1,1) & \leftrightarrow & (2,2,2) \\ (1,1,1,1,1,1) & \leftrightarrow & (2,1,1,1,1) \end{array}$$

The partitions of 6 into an even number of parts are listed on the left, and those having an odd number of parts are on the right. The pairing is given by the involution φ, and we see that the number of partitions into distinct parts is $\mathrm{Even}(6) - \mathrm{Odd}(6) = 1$.