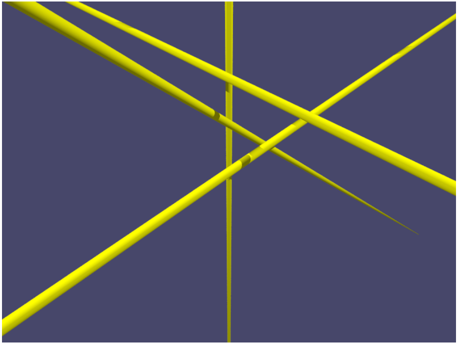
How to Park your Car on a Moduli Space

Maria Gillespie, Colorado State University
On joint work with Renzo Cavalieri (CSU) and Leonid Monin (University of Toronto)

University of Colorado Boulder, Feb 11, 2020

Background: Enumerative geometry

- **Q1:** Given four lines $\ell_1, \ell_2, \ell_3, \ell_4$ in 'general position' in three-dimensional space, how many lines pass through all four of them?

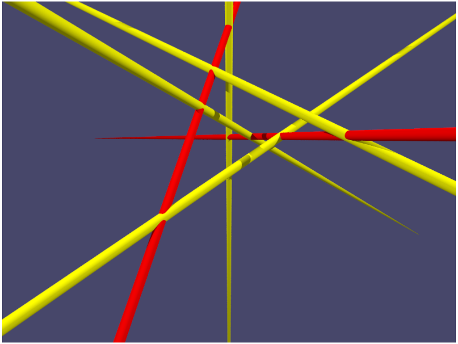


Background: Enumerative geometry

- **Q1:** Given four lines $\ell_1, \ell_2, \ell_3, \ell_4$ in 'general position' in three-dimensional space, how many lines pass through all four of them?

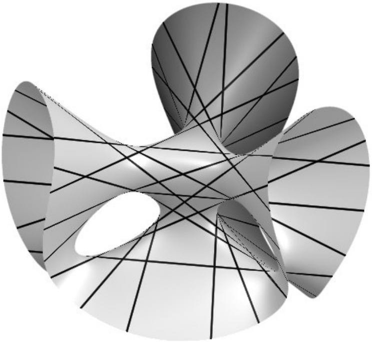
  A1: At most 2 (over $\mathbb{R}$).

  A2: Exactly 2 (over $\mathbb{C}$, in projective space).



Background: Enumerative geometry

- **Q2:** How many lines lie on a generic cubic surface in $\mathbb{CP}^3$?
  A: 27



Solving Enumerative Geometry Problems in Three Easy Steps!

Solving Enumerative Geometry Problems in Three Easy Steps!

1. Make the problem more difficult.

Solving Enumerative Geometry Problems in Three Easy Steps!

1. Make the problem more difficult.
2. Make it even harder.

Solving Enumerative Geometry Problems in Three Easy Steps!

1. Make the problem more difficult.
2. Make it even harder.
3. Turn it into a combinatorics problem, which may or may not be easier to solve than the original geometry question.

Solving Enumerative Geometry Problems in Three Easy Steps!

1. Make the problem more difficult.
2. Make it even harder.
3. Turn it into a combinatorics problem, which may or may not be easier to solve than the original geometry question.

Fine Print:

1. **Rephrase the Geometry:** Turn it into an intersection problem about families of geometric objects in some larger 'moduli space'.
2. **Geometry → Algebra:** State the intersection problem algebraically (in terms of the 'cohomology ring' or 'chow ring' of the moduli space).
3. **Algebra → Combinatorics:** Determine the combinatorial structure of the algebraic space, and use it to phrase the intersection problem in terms of simple combinatorial objects. Solve.

Example of the Three-Step Method

- **Q1:** Given four lines $\ell_1, \ell_2, \ell_3, \ell_4$ in 'general position' in $\mathbb{CP}^3$, how many lines pass through all four of them?

Example of the Three-Step Method

- **Q1:** Given four lines $\ell_1, \ell_2, \ell_3, \ell_4$ in 'general position' in $\mathbb{CP}^3$, how many lines pass through all four of them?

1. **Rephrase the Geometry:**
   - $X_i := \{\text{lines passing through } \ell_i\}$.
   - $X_i \subseteq \mathrm{Gr}$, where Gr is the moduli space of all lines in $\mathbb{CP}^3$.
   - Want to compute $|X_1 \cap X_2 \cap X_3 \cap X_4|$.

Example of the Three-Step Method

- **Q1:** Given four lines $\ell_1, \ell_2, \ell_3, \ell_4$ in 'general position' in $\mathbb{CP}^3$, how many lines pass through all four of them?

1. **Rephrase the Geometry:**
   - $X_i := \{\text{lines passing through } \ell_i\}$.
   - $X_i \subseteq \mathrm{Gr}$, where Gr is the moduli space of all lines in $\mathbb{CP}^3$.
   - Want to compute $|X_1 \cap X_2 \cap X_3 \cap X_4|$.
2. **Geometry $\rightarrow$ Algebra:**
   - $[X_1] = [X_2] = [X_3] = [X_4] \in H^*(\mathrm{Gr})$, multiplication in H^* corresponds to intersection (for generic representatives of each class).
   - Want to compute $[X_1]^4$.

Example of the Three-Step Method

- **Q1:** Given four lines $\ell_1, \ell_2, \ell_3, \ell_4$ in 'general position' in $\mathbb{CP}^3$, how many lines pass through all four of them?

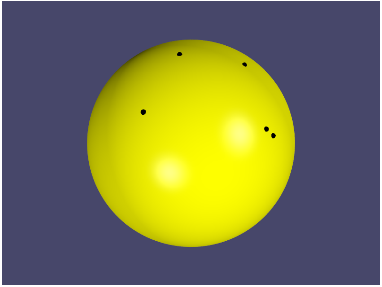
1. **Rephrase the Geometry:**
   - $X_i := \{\text{lines passing through } \ell_i\}$.
   - $X_i \subseteq \mathrm{Gr}$, where Gr is the moduli space of all lines in $\mathbb{CP}^3$.
   - Want to compute $|X_1 \cap X_2 \cap X_3 \cap X_4|$.
2. **Geometry → Algebra:**
   - $[X_1] = [X_2] = [X_3] = [X_4] \in H^*(\mathrm{Gr})$, multiplication in H^* corresponds to intersection (for generic representatives of each class).
   - Want to compute $[X_1]^4$.
3. **Algebra → Combinatorics:**
   - $H^*(\mathrm{Gr})$ is isomorphic to a quotient of the ring of symmetric functions
   - $[X_1]^4 = c \cdot [\mathrm{pt}]$ where $c = \#$ ways to fill a 2×2 grid with $1, 2, 3, 4$ with increasing rows and columns:

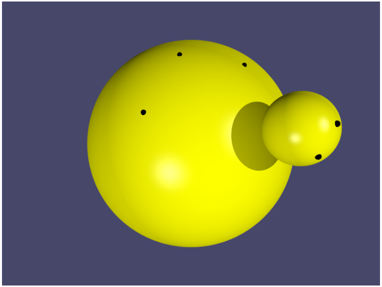
2	4
1	3

3	4
1	2

Moduli space of genus-0 curves

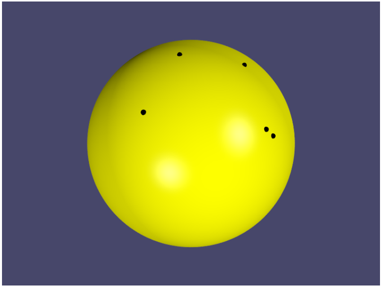
- $\overline{M}_{0,n}$ is space of (isom. classes of) complex 'stable curves' of genus 0 with n distinct 'marked points'.
- **Example:** $\mathbb{CP}^1$, need to mark at least 3 points for it to be 'stable': no nontrivial automorphisms.

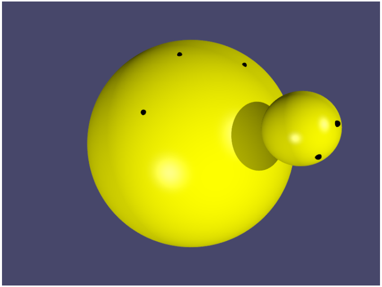




Moduli space of genus-0 curves

- $\overline{M}_{0,n}$ is space of (isom. classes of) complex 'stable curves' of genus 0 with n distinct 'marked points'.
- **Example:** $\mathbb{CP}^1$, need to mark at least 3 points for it to be 'stable': no nontrivial automorphisms.
- In general: Allow several copies of $\mathbb{CP}^1$ glued together at nodes, where each sphere has a total of at least 3 nodes and marked points.





How do you draw $\overline{M}_{0,n}$?

- $|\overline{M}_{0,3}| = 1$ - only one stable curve with 3 points up to isomorphism
- $\overline{M}_{0,4} \equiv \mathbb{P}^1$: fix 3 points, let the fourth vary.

How do you draw $\overline{M}_{0,n}$?

- $|\overline{M}_{0,3}| = 1$ - only one stable curve with 3 points up to isomorphism
- $\overline{M}_{0,4} \equiv \mathbb{P}^1$: fix 3 points, let the fourth vary.
- In general $\dim \overline{M}_{0,n+3} = n$, but not isomorphic to $\mathbb{P}^n$.
- **Kapranov map**: $\overline{M}_{0,n+3} \to \mathbb{P}^n$

How do you draw $\overline{M}_{0,n}$?

- $|\overline{M}_{0,3}| = 1$ - only one stable curve with 3 points up to isomorphism
- $\overline{M}_{0,4} \equiv \mathbb{P}^1$: fix 3 points, let the fourth vary.
- In general $\dim \overline{M}_{0,n+3} = n$, but not isomorphic to $\mathbb{P}^n$.
- **Kapranov map**: $\overline{M}_{0,n+3} \to \mathbb{P}^n$
- **Forgetting map**: $\overline{M}_{0,n+3} \to \overline{M}_{0,n+2}$ by forgetting last marked point (and stabilizing)
- **(Keel, Tevelev)**: Combining Kapranov, forgetting gives embedding

$$\overline{M}_{0,n+3} \hookrightarrow \overline{M}_{0,n+2} \times \mathbb{P}^n$$

How do you draw $\overline{M}_{0,n}$?

- $|\overline{M}_{0,3}| = 1$ - only one stable curve with 3 points up to isomorphism
- $\overline{M}_{0,4} \equiv \mathbb{P}^1$: fix 3 points, let the fourth vary.
- In general $\dim \overline{M}_{0,n+3} = n$, but not isomorphic to $\mathbb{P}^n$.
- **Kapranov map**: $\overline{M}_{0,n+3} \to \mathbb{P}^n$
- **Forgetting map**: $\overline{M}_{0,n+3} \to \overline{M}_{0,n+2}$ by forgetting last marked point (and stabilizing)
- **(Keel, Tevelev)**: Combining Kapranov, forgetting gives embedding

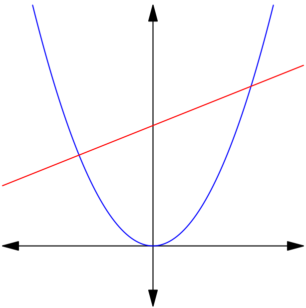
$$\overline{M}_{0,n+3} \hookrightarrow \overline{M}_{0,n+2} \times \mathbb{P}^n$$

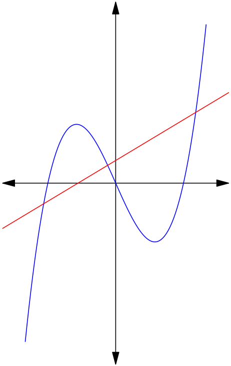
- Iterate: Get embedding

$$\overline{M}_{0,n+3} \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^2 \times \cdots \times \mathbb{P}^n$$

Degree of an embedding: How ugly is your drawing?

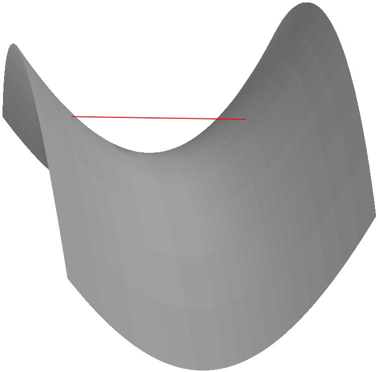
- Suppose $X \hookrightarrow \mathbb{P}^n$ and $\dim(X) = d$. Then the **degree** of this embedding is the number of points in an intersection of X with d generic hyperplanes in $\mathbb{P}^n$.





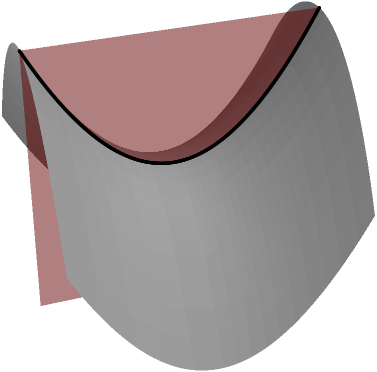
Degree of an embedding: How ugly is your drawing?

- Suppose $X \hookrightarrow \mathbb{P}^n$ and $\dim(X) = d$. Then the **degree** of this embedding is the number of points in an intersection of X with d generic hyperplanes in $\mathbb{P}^n$.



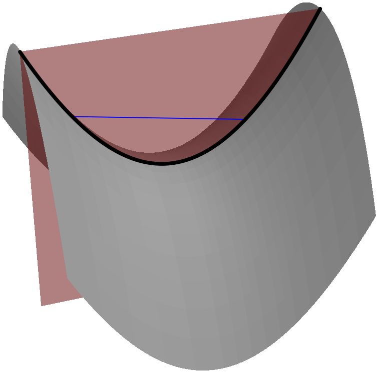
Degree of an embedding: How ugly is your drawing?

- Suppose $X \hookrightarrow \mathbb{P}^n$ and $\dim(X) = d$. Then the **degree** of this embedding is the number of points in an intersection of X with d generic hyperplanes in $\mathbb{P}^n$.



Degree of an embedding: How ugly is your drawing?

- Suppose $X \hookrightarrow \mathbb{P}^n$ and $\dim(X) = d$. Then the **degree** of this embedding is the number of points in an intersection of X with d generic hyperplanes in $\mathbb{P}^n$.



Multidegree: How ugly is your 'multi-drawing'?

- Suppose $X \hookrightarrow \mathbb{P}^1 \times \cdots \times \mathbb{P}^n$. Then the $(k_1, \ldots, k_n)$-**multidegree**,

$$\deg_{k_1,\ldots,k_n}(X),$$

is the expected size of intersection of X with a total of $k_1 + \cdots + k_n$ hyperplanes, k_i of which are from $\mathbb{P}^i$ for each i, where $k_1 + \cdots + k_n = d$.

Multidegree: How ugly is your 'multi-drawing'?

- Suppose $X \hookrightarrow \mathbb{P}^1 \times \cdots \times \mathbb{P}^n$. Then the $(k_1, \ldots, k_n)$-**multidegree**,

$$\deg_{k_1,\ldots,k_n}(X),$$

  is the expected size of intersection of X with a total of $k_1 + \cdots + k_n$ hyperplanes, k_i of which are from $\mathbb{P}^i$ for each i, where $k_1 + \cdots + k_n = d$.

- **Total degree** (Van der Waarden:) Let C be projectivization of preimage of X in affine space $\mathbb{A}^2 \times \cdots \times \mathbb{A}^{n+1} = \mathbb{A}^{(n+1)(n+2)/2-1}$. Then

$$\deg(C) = \sum_{(k_1,\ldots,k_n)} \deg_{k_1,\ldots,k_n} X.$$

Multidegree: How ugly is your ‘multi-drawing’?

- Suppose $X \hookrightarrow \mathbb{P}^1 \times \cdots \times \mathbb{P}^n$. Then the $(k_1, \ldots, k_n)$-**multidegree**,

$$\deg_{k_1,\ldots,k_n}(X),$$

  is the expected size of intersection of X with a total of $k_1 + \cdots + k_n$ hyperplanes, k_i of which are from $\mathbb{P}^i$ for each i, where $k_1 + \cdots + k_n = d$.
- **Total degree** (Van der Waarden:) Let C be projectivization of preimage of X in affine space $\mathbb{A}^2 \times \cdots \times \mathbb{A}^{n+1} = \mathbb{A}^{(n+1)(n+2)/2-1}$. Then

$$\deg(C) = \sum_{(k_1,\ldots,k_n)} \deg_{k_1,\ldots,k_n} X.$$

- **Geometry → Algebra:** Multidegrees $\deg_{(k_1,\ldots,k_n)}(\overline{M}_{0,n+3})$ satisfy a simple recursion in terms of $k_1, \ldots, k_n$

Multidegree: How ugly is your 'multi-drawing'?

- Suppose $X \hookrightarrow \mathbb{P}^1 \times \cdots \times \mathbb{P}^n$. Then the $(k_1, \ldots, k_n)$-**multidegree**,
$$\deg_{k_1,\ldots,k_n}(X),$$
is the expected size of intersection of X with a total of $k_1 + \cdots + k_n$ hyperplanes, k_i of which are from $\mathbb{P}^i$ for each i, where $k_1 + \cdots + k_n = d$.
- **Total degree** (Van der Waarden:) Let C be projectivization of preimage of X in affine space $\mathbb{A}^2 \times \cdots \times \mathbb{A}^{n+1} = \mathbb{A}^{(n+1)(n+2)/2-1}$. Then
$$\deg(C) = \sum_{(k_1,\ldots,k_n)} \deg_{k_1,\ldots,k_n} X.$$
- **Geometry → Algebra:** Multidegrees $\deg_{(k_1,\ldots,k_n)}(\overline{M}_{0,n+3})$ satisfy a simple recursion in terms of $k_1, \ldots, k_n$
- **Algebra → Combinatorics:** Computationally, observe
$$\sum_{k_1,\ldots,k_n} \deg_{k_1,\ldots,k_n}(\overline{M}_{0,n+3}) = (2n-1)!!!!!!!$$

Multidegree: How ugly is your 'multi-drawing'?

- Suppose $X \hookrightarrow \mathbb{P}^1 \times \cdots \times \mathbb{P}^n$. Then the $(k_1, \ldots, k_n)$-**multidegree**,
$$\deg_{k_1,\ldots,k_n}(X),$$
is the expected size of intersection of X with a total of $k_1 + \cdots + k_n$ hyperplanes, k_i of which are from $\mathbb{P}^i$ for each i, where $k_1 + \cdots + k_n = d$.
- **Total degree** (Van der Waarden:) Let C be projectivization of preimage of X in affine space $\mathbb{A}^2 \times \cdots \times \mathbb{A}^{n+1} = \mathbb{A}^{(n+1)(n+2)/2-1}$. Then
$$\deg(C) = \sum_{(k_1,\ldots,k_n)} \deg_{k_1,\ldots,k_n} X.$$
- **Geometry → Algebra:** Multidegrees $\deg_{(k_1,\ldots,k_n)}(\overline{M}_{0,n+3})$ satisfy a simple recursion in terms of $k_1, \ldots, k_n$
- **Algebra → Combinatorics:** Computationally, observe
$$\sum_{k_1,\ldots,k_n} \deg_{k_1,\ldots,k_n}(\overline{M}_{0,n+3}) = (2n-1)!!$$

How to Park your Car (in 1966)

"**A parking problem—the case of the capricious wives.** *Let st. be a street with p parking places. A car occupied by a man and his dozing wife enters st. at the left and moves towards the right. The wife awakens at a capricious moment and orders her husband to park immediately! He dutifully parks at his present location, if it is empty, and if not, continues to the right and parks at the next available space. Suppose st. to be initially empty and c cars arrive with independently capricious wives in each car. What is the probability that they all find parking places?*"

–Konheim and Weiss, 1966

How to Park your Car (Modern-day version)

- **Setup:** n cars $1, 2, \ldots, n$ lined up to enter parking lot in that order. n parking spaces $p_1, p_2, \ldots, p_n$ in order.
- Each car has a **preferred spot** to which they drive; if it's taken, they take the next available space. Each only enters parking lot once.

How to Park your Car (Modern-day version)

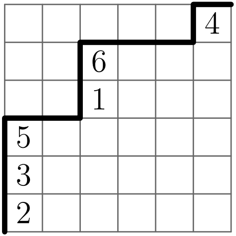
- **Setup:** n cars $1, 2, \ldots, n$ lined up to enter parking lot in that order. n parking spaces $p_1, p_2, \ldots, p_n$ in order.
- Each car has a **preferred spot** to which they drive; if it's taken, they take the next available space. Each only enters parking lot once.
- **Example:** Cars $2, 3, 5$ prefer spot p_1; $1, 6$ prefer p_3; 4 prefers p_6

How to Park your Car (Modern-day version)

- **Setup:** n cars $1, 2, \ldots, n$ lined up to enter parking lot in that order. n parking spaces $p_1, p_2, \ldots, p_n$ in order.
- Each car has a **preferred spot** to which they drive; if it's taken, they take the next available space. Each only enters parking lot once.
- **Example:** Cars $2, 3, 5$ prefer spot p_1; $1, 6$ prefer p_3; 4 prefers p_6
- **Parking Function**: A preference function $\{1, \ldots, n\} \to \{p_1, \ldots, p_n\}$ such that all cars end up parked.

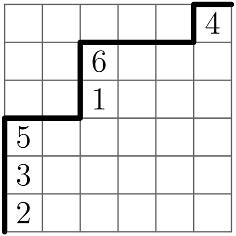
How to Park your Car (Modern-day version)

- **Setup:** n cars $1, 2, \ldots, n$ lined up to enter parking lot in that order. n parking spaces $p_1, p_2, \ldots, p_n$ in order.
- Each car has a **preferred spot** to which they drive; if it's taken, they take the next available space. Each only enters parking lot once.
- **Example:** Cars $2, 3, 5$ prefer spot p_1; $1, 6$ prefer p_3; 4 prefers p_6
- **Parking Function**: A preference function $\{1, \ldots, n\} \to \{p_1, \ldots, p_n\}$ such that all cars end up parked.
- Draw as **labeled Dyck path**; must stay above diagonal:



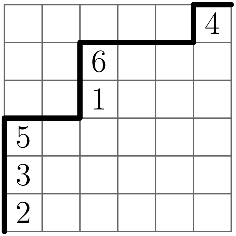
Column-restricted parking functions and $(2n-1)!!$

- **Dominance Index**: $D(i)$ of a car i is the number of columns to its left that contain no entry greater than i.



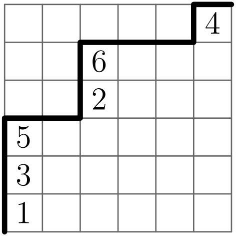
Column-restricted parking functions and $(2n-1)!!$

- **Dominance Index**: $D(i)$ of a car i is the number of columns to its left that contain no entry greater than i.
- A parking function is **column-restricted** if for every car i, $D(i) < i$.



Column-restricted parking functions and $(2n-1)!!$

- **Dominance Index**: $D(i)$ of a car i is the number of columns to its left that contain no entry greater than i.
- A parking function is **column-restricted** if for every car i, $D(i) < i$.



Column-restricted parking functions and $(2n-1)!!$

- **Dominance Index**: $D(i)$ of a car i is the number of columns to its left that contain no entry greater than i.
- A parking function is **column-restricted** if for every car i, $D(i) < i$.

Theorem (Cavalieri, G., Monin)

The multidegree $\deg_{k_1,\ldots,k_n}(\overline{M}_{0,n+3})$ *is equal to the number of column-restricted parking functions with column heights* $k_1, \ldots, k_n$.

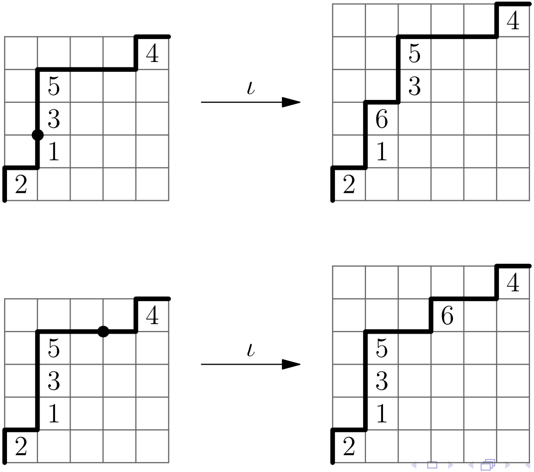
Theorem (Cavalieri, G., Monin)

There are a total of $(2n-1)!!$ *column-restricted parking functions of height* n.

Corollary: The total degree of the embedding of $\overline{M}_{0,n+3}$ into $\mathbb{P}^1 \times \cdots \times \mathbb{P}^n$ is $(2n-1)!!$.

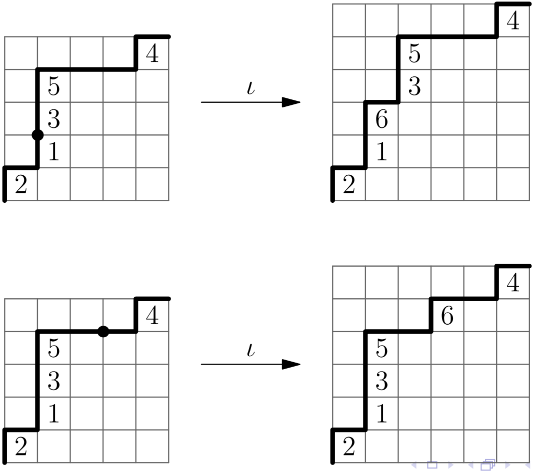
Idea for proof of enumeration by (2n-1)!!

- $\mathrm{CPF}(n)$: Number of column-restricted parking functions of size n
- Want to show: $\mathrm{CPF}(n) = (2n-1)\mathrm{CPF}(n-1)$

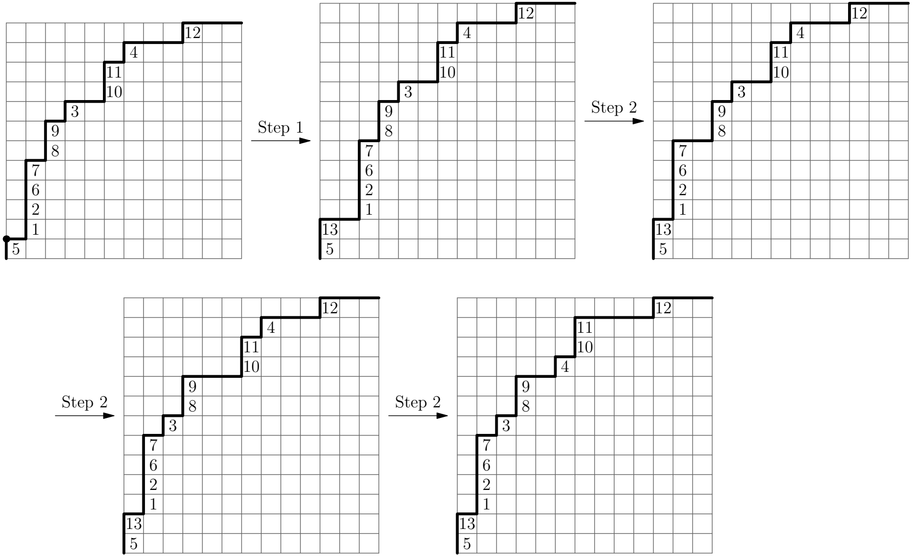


Idea for proof of enumeration by (2n-1)!!

- $\mathrm{CPF}(n)$: Number of column-restricted parking functions of size n
- Want to show: $\mathrm{CPF}(n) = (2n-1)\mathrm{CPF}(n-1)$
- $2n-1$ points on any Dyck path of size $n-1$; "insert" n at each point.



Idea for proof of enumeration by (2n-1)!!



Reference

- R. Cavalieri, M. Gillespie, L. Monin, Projective embeddings of $\overline{M}_{0,n}$ and Parking Functions, arxiv:1808.03573.

THANK YOU!