# The solution to the partition reconstruction problem

Maria Monks
monks@mit.edu

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**Abstract**

Given a partition $\lambda$ of $n$, a *$k$-minor* of $\lambda$ is a partition of $n-k$ whose Young diagram fits inside that of $\lambda$. We find an explicit function $g(n)$ such that any partition of $n$ can be reconstructed from its set of $k$-minors if and only if $k \leq g(n)$. In particular, partitions of $n \geq k^2+2k$ are uniquely determined by their sets of $k$-minors. This result completely solves the partition reconstruction problem and also a special case of the character reconstruction problem for finite groups.

## 1 Introduction

The problem of partition reconstruction can be stated as follows. For any positive integer $k$, define a $k$-minor of a partition $\lambda$ of a positive integer $n > k$ to be a partition of $n-k$ whose Young diagram fits inside that of $\lambda$. It is natural to ask for which $n$ and $k$ we can uniquely determine any partition of $n$ from its set of $k$-minors.

In this paper, we demonstrate that partitions of any positive integer $n \geq 2$ other than 5, 12, 21, and 32 can be reconstructed from their $k$-minors if and only if $k \leq \min_{0\leq t\leq n} \rho(n-t+2)+t-2$, where $\rho(m)$ is the smallest positive divisor $d$ of $m$ for which $d \geq \sqrt{m}$. This result is verified by computer for all $n < 1765$ and proven for all $n \geq 1765$. For $n = 5$, 12, 21, or 32, the partitions of $n$ can be reconstructed from their $k$-minors if and only if $k$ is at most 1, 3, 5, or 7 respectively. Together, these results solve this reconstruction problem completely.

Pretzel and Siemons [8] demonstrated that partitions of $n$ can be reconstructed from their sets of $k$-minors if $n \geq 2k^2+8k+6$, and asked whether this bound is the best possible. In fact, it is not: we can improve this result to $n \geq k^2+2k$. This bound is the best possible, in the sense that for every $k$ there exist two distinct partitions of $k^2+2k-1$ which have the same set of $k$-minors.

The problem of partition reconstruction arises naturally in representation theory. The character reconstruction problem for finite groups [8] asks when we can uniquely recover the character of a representation of a finite group $G$ over a field of characteristic zero from its restriction to various subgroups. In Section 4.1, we show that our results on partition reconstruction solve this problem when $G$ is the symmetric group $S_n$ acting on $\{1, 2, \ldots, n\}$, and the subgroup is the stabilizer of any subset of $\{1, 2, \ldots, n\}$.

Reconstruction of partitions also has applications to related reconstruction problems. Define a cycle $k$-minor of a permutation $p \in S_n$ to be a permutation in $S_{n-k}$ formed by deleting $k$ elements of the decomposition of $p$ into disjoint cycles and re-numbering the remaining entries from 1 to $n-k$, preserving the relative order of the entries. The problem of reconstructing a permutation from its set of cycle $k$-minors is currently open [7]. In Section 4.2, we demonstrate that for any $k$, we can reconstruct the conjugacy class of a permutation in $S_n$ from its cycle $k$-minors for sufficiently large $n$.

## 1.1 Notation

We now introduce the definitions needed to state the main results. Further notation will be provided as needed in Section 3.

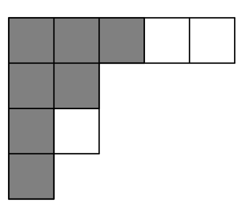
Let $n$ be a positive integer. A *partition* $\lambda$ *of* $n$ is an array $[\lambda_1, \lambda_2, \ldots, \lambda_m]$ of positive integers which satisfy $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$ and $\sum_{i=1}^{m} \lambda_i = n$. If $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_m]$ is a partition of $n$, we say that $n$ is the *size* of $\lambda$, and we call $\lambda_1, \lambda_2, \ldots, \lambda_m$ the *parts* of $\lambda$. For any partition $\lambda$, we will always use $\lambda_1$ to denote the largest part, $\lambda_2$ the second largest, and so on, and we define $\lambda_j = 0$ for any $j$ larger than the number of parts of $\lambda$. We now introduce the notion of a minor of a partition.

**Definition.** Let $\lambda$ be a partition of $n$, and let $\mu$ be a partition of $n-k$. Then $\mu$ is a *$k$-minor* of $\lambda$ if $\mu_i \leq \lambda_i$ for all $i$. We write $M_k(\lambda)$ to denote the set of all $k$-minors of $\lambda$.

The *Young diagram* of a partition $\lambda = [\lambda_1, \ldots, \lambda_m]$ is a partial grid of squares consisting of $m$ rows, aligned at the left, with the $i$th row containing $\lambda_i$ squares for each $i \leq m$. Henceforth, we will refer to a partition and its Young diagram interchangeably.

A *corner square* of the Young diagram of a partition $\lambda$ is a square $X$ for which there are no squares directly below or directly to the right of $X$. We write $\lambda/X$ to denote the partition whose Young diagram is formed by removing $X$ from $\lambda$. In general, a $k$-minor of $\lambda$ can be formed by iterating $k$ times the process of removing a single corner square, beginning with the Young diagram of $\lambda$. We also write $\lambda/\mu$ to denote the set of squares in a partition $\lambda$ that are not in its minor $\mu$.

For example, consider the partition $\lambda = [5, 2, 2, 1]$. The Young diagram of $\lambda$ is shown below. The Young diagram of its 2-minor $\mu = [3, 2, 1, 1]$ is shaded, and we see that the Young diagram of $\mu$ fits inside that of $\lambda$.



Continuing with this example, we find that

$$M_3([5,2,2,1]) = \{[5,2],[5,1,1],[4,2,1],[4,1,1,1],[3,2,2],[3,2,1,1],[2,2,2,1]\}.$$

We wish to find the pairs of integers $n$ and $k$ for which $M_k(\lambda) = M_k(\nu)$ implies that $\lambda = \nu$ for all partitions $\lambda$ and $\nu$ of $n$. If this property is satisfied for a given $n$ and $k$, we say *reconstructibility holds*, and otherwise it *fails*. The partition reconstruction problem asks when reconstructibility holds. To answer this, we require the following number theoretic function.

**Definition.** For any positive integer $m$, let $\rho(m)$ be the smallest divisor $d$ of $m$ for which $d \geq \sqrt{m}$.

# 2 Main Results

In this section we state the main results and defer the proofs until section 3.

**Theorem 2.1.** *Let $n$ and $k$ be positive integers with $k < n$. For $n \notin \{5, 12, 21, 32\}$, define*

$$g(n) = \min_{0 \leq t \leq n} \rho(n+2-t) - 2 + t$$

*and also define $g(5) = 1$, $g(12) = 3$, $g(21) = 5$, $g(32) = 7$. Then partitions of $n$ can be reconstructed from their sets of $k$-minors if and only if $k \leq g(n)$.*

Theorem 2.1 provides us with an efficient means of determining whether reconstructibility holds for a given $n$ and $k$ by finding the minimum of a set of only $n$ values. In fact, there are even more efficient ways of computing $g(n)$, as the following two theorems show.

**Theorem 2.2.** *Let $n > 2$ be a positive integer other than 5, 12, 21, and 32. Then*

$$g(n) = \min\{\rho(n+2) - 2, g(n-1) + 1\}. \tag{2.1}$$

Furthermore, we have the following explicit formula when $n$ is two less than a square.

**Theorem 2.3.** *Suppose $n + 2$ is a perfect square. Then $g(n) = \sqrt{n+2} - 2$.*

Theorems 2.2 and 2.3 enable us to compute $g(n)$ without computing all values of $\rho(n+2-t) - 2 + t$ for $0 \leq t \leq n$. Given $n$, we can first find the largest $m \leq n$ for which $m + 2$ is a perfect square, compute $g(m)$ using Theorem 2.3, and then use (2.1) to compute $g(m+1), g(m+2), \ldots, g(n)$. For example, $g(63) = \min\{\rho(65) - 2, g(62) + 1\} = \min\{13 - 2, \sqrt{64} - 2 + 1\} = 7$.

Theorem 2.3 can be extended to several other infinite families of positive integers $n$. For instance, if $d$ is a fixed positive integer and $n = r(r + d) - 2$ for some positive integer $r$, then as long as $r$ is sufficiently large compared to $d$ we have $g(n) = \rho(n + 2) - 2$. The proof of this fact is similar to that of Theorem 2.3, and we omit it.

The order of growth of $g$ is approximately $\sqrt{n}$, as the following theorem illustrates.

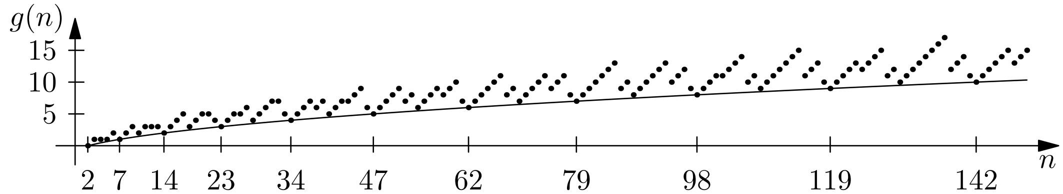



Figure 1: The plot of $g(n)$ for $2 \leq n \leq 150$, along with the lower bound of $\sqrt{n+2} - 2$. The lower bound is achieved when $n + 2$ is a perfect square, as indicated.

**Theorem 2.4.** *For all positive integers $n \geq 2$,*

$$\sqrt{n+2} - 2 \leq g(n) \leq \sqrt{n+2} + 3\sqrt[4]{n+2}.$$

Hence, for large $n$ we can approximate $g(n)$ as $\sqrt{n+2}$. This is usually unnecessary due to the formulas above, but it provides useful intuition about the values of $g$.

While $g$ is clearly not an invertible function (see Figure 1), we can provide a tight bound on reconstructibility for a fixed positive integer $k$.

**Theorem 2.5.** *Let $k$ be a positive integer. Then reconstructibility holds for $n \geq k^2 + 2k$, and fails for $n = k^2 + 2k - 1$.*

Notice that there are some values of $n$ that are *less* than $k^2 + 2k - 1$ for which any partition of size $n$ can be reconstructed from its set of $k$-minors. For example, for $k = 6$, reconstructibility holds for $n = 27, 30, 31, 32, 36, 37, 38, 39, 41, 42, 43, 44, 45, 46$. It fails for $n = 47$, but for $n \geq 6^2 + 2 \cdot 6 = 48$, partitions of $n$ can always be reconstructed from their 6-minors.

# 3 Proofs

We first prove Theorem 2.2, which we restate below.

**Theorem 2.2.** Let $n > 2$ be a positive integer other than 5, 12, 21, and 32. Then

$$g(n) = \min\{\rho(n+2) - 2, g(n-1) + 1\}.$$

*Proof.* It can be verified by a direct calculation that this recursion holds for $n = 6$, 13, 22, and 33. Suppose $n \notin \{5, 6, 12, 13, 21, 22, 32, 33\}$. Then by the definition of $g$,

$$\begin{aligned} g(n) &= \min_{0 \leq t \leq n} \rho(n+2-t) - 2 + t \\ &= \min_{-1 \leq t \leq n-1} \rho(n+2-(t+1)) - 2 + (t+1) \\ &= \min\left\{\rho(n+2) - 2, \left(\min_{0 \leq t \leq n-1} \rho((n-1)+2-t) - 2 + t\right) + 1\right\} \\ &= \min\{\rho(n+2) - 2, g(n-1) + 1\} \end{aligned}$$

as desired. $\square$

We proceed to prove the lower bound of Theorem 2.4.

**Lemma 3.1.** *For all positive integers $n \geq 2$,*

$$\sqrt{n+2} - 2 \leq g(n).$$

*Proof.* A straightforward calculation shows that the inequality holds for $n \leq 32$ (see Figure 1). Suppose $n \geq 33$. Then by the definition of $g$, we have $g(n) = \rho(n+2-t) - 2 + t$ for some $t$ such that $0 \leq t \leq n$. Thus

$$\begin{aligned} g(n) &= \rho(n+2-t) - 2 + t \\ &\geq \sqrt{n+2-t} - 2 + t \\ &\geq \sqrt{n+2} - 2 \end{aligned}$$

where the final inequality follows from the fact that $\sqrt{n+2-t} - 2 + t$ is an increasing function of $t$ when $0 \leq t \leq n$. □

Theorem 2.3 now follows.

**Theorem 2.3.** Suppose $n+2$ is a perfect square. Then $g(n) = \sqrt{n+2} - 2$.

*Proof.* Suppose $n+2$ is a perfect square. Then $\rho(n+2) - 2 = \sqrt{n+2} - 2$, so $g(n) \leq \sqrt{n+2} - 2$ by the definition of $g$. Furthermore, from Lemma 3.1 we have $g(n) \geq \sqrt{n+2} - 2$, and hence $g(n) = \sqrt{n+2} - 2$. □

We now prove the upper bound of Theorem 2.4.

**Lemma 3.2.** *For all positive integers $n \geq 2$,*

$$g(n) \leq \sqrt{n+2} + 3\sqrt[4]{n+2}.$$

*Proof.* Straightforward computation shows that the bound holds for $n \leq 32$. Suppose $n \geq 33$, so that the recursion (2.1) holds. Note that for any positive integers $a$ and $r$ for which $r^2 - a^2 - 2 \geq 2$, we have

$$\begin{aligned} g(r^2 - a^2 - 2) &\leq \rho(r^2 - a^2) - 2 \\ &= \rho((r-a)(r+a)) - 2 \\ &\leq r + a - 2 \end{aligned}$$

by the definitions of $g$ and $\rho$, and similarly

$$g(r^2 - r - a(a+1) - 2) \leq r + a - 2$$

whenever $r^2 - r - a(a+1) - 2 \geq 2$. Furthermore, iterating the inequality $g(m+1) \leq g(m) + 1$ (a consequence of Theorem 2.2), we have

$$g(m+t) \leq g(m) + t$$

for all $t \geq 0$ and $m \geq 2$. Combining these, we obtain the following two inequalities.

$$g(r^2 - a^2 - 2 + t) \leq r + a - 2 + t \tag{3.1}$$

$$g(r^2 - r - a(a+1) - 2 + t) \leq r + a - 2 + t \tag{3.2}$$

Now, let $r = \lceil \sqrt{n+2} \rceil$, so that $(r-1)^2 + 1 \leq n + 2 \leq r^2$.

**Case 1.** Suppose $(r-1)^2 + 1 \leq n + 2 \leq r^2 - r - 1$. Let $a$ be the smallest positive integer such that $r^2 - r - a(a+1) - 2 \leq n$. Then

$$n = r^2 - r - a(a+1) - 2 + t$$

for some $t \leq a(a+1) - (a-1)a - 1 = 2a - 1$. In addition, by the definition of $a$ we have $n \leq r^2 - r - (a-1)a - 2 - 1$. Since $(r-1)^2 - 1 \leq n$, we have $(r-1)^2 - 1 \leq r^2 - r - (a-1)a - 2 - 1$, which we can solve for $a$ to obtain

$$a \leq \frac{1}{2} + \sqrt{r - \frac{11}{4}}.$$

Therefore, $t \leq 2a - 1 \leq 2\sqrt{r - \frac{11}{4}}$. By (3.2), we have

$$\begin{aligned} g(n) &= g(r^2 - r - a(a+1) - 2 + t) \\ &\leq r + a - 2 + t \\ &\leq r + 3\sqrt{r - \frac{11}{4}} - \frac{3}{2} \\ &= \left\lceil \sqrt{n+2} \right\rceil + 3\sqrt{\left\lceil \sqrt{n+2} \right\rceil - \frac{11}{4}} - \frac{3}{2} \\ &\leq \sqrt{n+2} + 3\sqrt[4]{n+2} \end{aligned}$$

as desired.

**Case 2.** Suppose $r^2 - r \leq n + 2 \leq r^2 - 1$. Let $a$ be the smallest positive integer such that $r^2 - a^2 - 2 \leq n$. Then $n = r^2 - a^2 - 2 + t$ for some $t \leq a^2 - (a-1)^2 - 1 = 2a - 2$. In addition, by the definition of $a$ we have $n \leq r^2 - (a-1)^2 - 3$. Since $r^2 - r - 2 \leq n$ as well, it follows that $r^2 - r - 2 \leq r^2 - (a-1)^2 - 3$, and so

$$a \leq \sqrt{r-1} + 1.$$

Therefore $t \leq 2a - 2 \leq 2\sqrt{r-1}$. By (3.1), we have

$$\begin{aligned} g(n) &= g(r^2 - a^2 - 2 + t) \\ &\leq r + a - 2 + t \\ &\leq r + 3\sqrt{r-1} - 1 \\ &= \left\lceil \sqrt{n+2} \right\rceil + 3\sqrt{\left\lceil \sqrt{n+2} \right\rceil - 1} - 1 \\ &\leq \sqrt{n+2} + 3\sqrt[4]{n+2} \end{aligned}$$

as desired.

**Case 3.** Suppose $n+2=r^2$. By Theorem 2.3 we have $g(n)=\sqrt{n+2}-2 \le \sqrt{n+2}+3\sqrt[4]{n+2}$.

Hence, $g(n) \le \sqrt{n+2}+3\sqrt[4]{n+2}$ for all $n$. □

Theorem 2.4, which we restate below, follows directly from Lemmas 3.1 and 3.2.

**Theorem 2.4.** For all positive integers $n \ge 2$,

$$\sqrt{n+2}-2 \le g(n) \le \sqrt{n+2}+3\sqrt[4]{n+2}.$$

To prove the remaining theorems, we first introduce some new terminology.

**Definition.** Let $\lambda$ and $\mu$ be any two partitions. The *union* of $\lambda$ and $\mu$ is the partition $\lambda \cup \mu$ whose $i$th part is $\max\{\lambda_i, \mu_i\}$ for all $i$. Similarly, the *intersection* of $\lambda$ and $\mu$ is the partition $\lambda \cap \mu$ whose $i$th part is $\min\{\lambda_i, \mu_i\}$ for all $i$.

In other words, the union or intersection of two partitions is formed by taking the union or intersection, respectively, of the sets of squares in their Young diagrams.

**Definition.** Let $X$ be a square of the Young diagram of a partition $\lambda$. Then the *outer region* of $X$, denoted $\operatorname{Out}_\lambda(X)$, is the set of all squares that lie strictly below or strictly to the right of $X$, and the *inner region* of $X$, denoted $\operatorname{In}_\lambda(X)$, is the rectangle of squares with corners at $X$ and the upper left hand corner of the diagram.

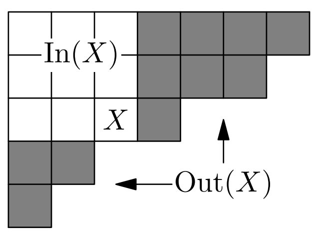



Figure 2: A square $X$, with its outer region shaded.

We often refer to the outer region or inner region of $X$ simply as $\operatorname{Out}(X)$ or $\operatorname{In}(X)$ when the partition in question is clear. Notice that $\operatorname{In}_\lambda(X)$ is always a minor of $\lambda$, and $\lambda/\operatorname{In}_\lambda(X)=\operatorname{Out}_\lambda(X)$.

**Lemma 3.3.** *Let $\lambda$ and $\nu$ be partitions of $n$ and let $\kappa=\lambda\cup\nu$. Then $M_k(\lambda)=M_k(\nu)$ if and only if $|\operatorname{Out}_\nu(X)| \le k-1$ for all squares $X \in \kappa/\lambda$ and $|\operatorname{Out}_\lambda(Y)| \le k-1$ for all $Y \in \kappa/\nu$.*

*Proof.* Suppose $M_k(\lambda)=M_k(\nu)$, and assume there exists a square $X \in \kappa/\lambda$ having $|\operatorname{Out}_\nu(X)| \ge k$. Then there exists a $k$-minor $\mu$ of $\nu$ that contains $X$, formed by removing $k$ corner squares in succession from $\operatorname{Out}_\lambda(X)$. Since $X$ is not in $\lambda$, the minor $\mu \in M_k(\nu)$ cannot be in $M_k(\lambda)$, which is a contradiction since $M_k(\lambda)=M_k(\mu)$. Similarly, if there exists a square $Y \in \kappa/\nu$ with $|\operatorname{Out}_\lambda(Y)| \geq k$, then there is a minor of $\lambda$ that is not a minor of $\nu$. Hence, $|\operatorname{Out}_\nu(X)| \leq k-1$ for all squares $X \in \kappa/\lambda$ and $|\operatorname{Out}_\lambda(Y)| \leq k-1$ for all $Y \in \kappa/\nu$.

Conversely, suppose $|\operatorname{Out}_\nu(X)| \leq k-1$ for all $X \in \kappa/\lambda$ and $|\operatorname{Out}_\lambda(Y)| \leq k-1$ for all $Y \in \kappa/\nu$. Let $\mu$ be a $k$-minor of $\lambda$, and let $X \in \kappa/\nu$ be arbitrary. Assume that $\mu$ contains the square $X$. Then $\mu$ contains $\operatorname{In}(X)$, and since $|\operatorname{In}(X)| + |\operatorname{Out}_\lambda(X)| = n$ we have $|\operatorname{In}(X)| > n-k$. Hence, $\mu$ contains more than $n-k$ squares, a contradiction. Since $X$ was arbitrary, it follows that $\mu$ cannot contain any square in $\kappa/\nu$, and so $\mu$ is a $k$-minor of $\nu$ as well. By a similar argument, any $k$-minor $\mu$ of $\nu$ is a minor of $\lambda$, and so $M_k(\lambda) = M_k(\nu)$. □

We now introduce a metric on partitions.

**Definition.** Let $\lambda$ and $\nu$ be any two partitions. Then the *distance* between $\lambda$ and $\nu$, denoted $d(\lambda, \nu)$, is given by

$$\sum_{i=1}^{\infty} |\lambda_i - \nu_i|.$$

Alternatively, the distance between $\lambda$ and $\nu$ is the number of squares that appear in the Young diagram of either $\lambda$ or $\nu$ but not in both. This yields the identity

$$d(\lambda, \nu) = |\lambda \cup \nu| - |\lambda \cap \nu|. \tag{3.3}$$

Notice that if $\lambda$ and $\nu$ are partitions of the same size and $d(\lambda, \nu) = 2$, then $\lambda \cup \nu$ has exactly one corner square that is in $\nu$ but not in $\lambda$ and exactly one corner square that is in $\lambda$ but not in $\nu$. Thus we obtain the following corollary to Lemma 3.3.

**Corollary 3.4.** *Let $\lambda$ and $\nu$ be any two partitions of $n$ having $d(\lambda, \nu) = 2$. Let $X$ and $Y$ be the unique corner squares of $\lambda \cup \nu$ that are not in $\lambda$ and $\nu$, respectively. Then $M_k(\lambda) = M_k(\nu)$ if and only if each of $|\operatorname{Out}_\nu(X)|$ and $|\operatorname{Out}_\lambda(Y)|$ is at most $k-1$.*

We now show that if reconstructibility fails for $n$ and $k$, there are two partitions $\lambda$ and $\nu$ of $n$ with $d(\lambda, \nu) = 2$ that have the same set of $k$-minors.

**Lemma 3.5.** *Let $k$ be a positive integer, and suppose $n$ is a positive integer for which there are partitions $\lambda \neq \mu$ of $n$ for which $M_k(\lambda) = M_k(\mu)$. Then there exists a partition $\nu$ of $n$ such that $d(\lambda, \nu) = 2$ and $M_k(\lambda) = M_k(\nu)$.*

*Proof.* First note that since $\lambda \neq \mu$ we have $d(\lambda, \mu) > 0$. Also, by (3.3) and the inclusion-exclusion principle, we see that $d(\lambda, \mu) = |\lambda \cup \nu| - |\lambda \cap \nu| = |\lambda| + |\nu| - 2|\lambda \cap \nu| = 2n - 2|\lambda \cap \nu|$ is even, so $d(\lambda, \mu) \neq 1$. Hence $d(\lambda, \mu) \geq 2$.

We now construct $\nu$ as follows. The sizes of $\lambda$ and $\mu$ are equal and $d(\lambda, \mu) \geq 2$, so there must exist indices $s$ and $t$ such that $\mu_s < \lambda_s$ and $\lambda_t < \mu_t$. We can assume without loss of generality that $s < t$. Hence, $\mu_t \leq \mu_s$ and so $\lambda_t + 2 \leq \lambda_s$. Let $\sigma$ be the largest index such that $\lambda_\sigma = \lambda_s$, and let $\tau$ be the smallest index such that $\lambda_\tau = \lambda_t$. Notice that we have defined $\sigma$ and $\tau$ so that row $\lambda_\sigma$ contains a corner square, and adding a square to row $\lambda_\tau$

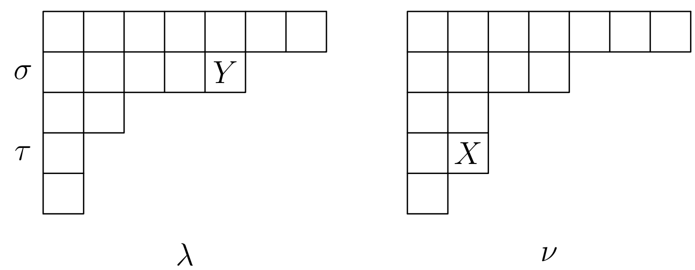
will result in a new partition. We also have $\mu_\sigma < \lambda_\sigma$ and $\lambda_\tau < \mu_\tau$. Since $\lambda_\tau + 2 \leq \lambda_\sigma$, it follows that we can move a square in the Young diagram from the part $\lambda_\sigma$ to $\lambda_\tau$ to form a new partition.

Let $m$ be the number of parts of $\lambda$, and let $\nu$ be the partition

$$[\lambda_1, \lambda_2, \ldots, \lambda_\sigma - 1, \ldots, \lambda_\tau + 1, \ldots, \lambda_m]$$

formed by moving a square of the Young diagram from the part $\lambda_\sigma$ to $\lambda_\tau$. Notice that $d(\lambda, \nu) = 2$. We proceed to show that $M_k(\lambda) = M_k(\nu)$.

Let $X$ be the square in row $\tau$ that is in $\nu$ but not in $\lambda$, and let $Y$ be the square in row $\sigma$ that is in $\lambda$ but not in $\nu$ as shown below. By Corollary 3.4, it suffices to show that each of $|\operatorname{Out}_\nu(X)|$ and $|\operatorname{Out}_\lambda(Y)|$ is at most $k-1$. Notice that $Y$ is not in $\mu$ since $\mu_\sigma < \lambda_\sigma$. Since $M_k(\lambda) = M_k(\mu)$, it follows from Lemma 3.3 that $|\operatorname{Out}_\lambda(Y)| \leq k-1$. Also, since $\lambda_\tau < \mu_\tau$, the square $X$ is in $\mu$ but not in $\lambda$, so $|\operatorname{Out}_\mu(X)| \leq k-1$. Furthermore, $X$ is in the same row and column in $\mu$ as it is in $\nu$, so $|\operatorname{In}_\mu(X)| = |\operatorname{In}_\nu(X)|$. Since $\mu$ and $\nu$ have the same size $n$, it follows that $|\operatorname{Out}_\nu(X)| = |\operatorname{Out}_\mu(X)| \leq k-1$ as desired. □




We now provide a necessary and sufficient condition for reconstructibility to hold for a given $n$ and $k$. We write $c \bmod a$ to denote the remainder when $c$ is divided by $a$.

**Lemma 3.6.** *Let $n$ and $k$ be positive integers. Then there exist partitions $\lambda \neq \mu$ of $n$ with $M_k(\lambda) = M_k(\mu)$ if and only if $n$ can be expressed in the form*

$$n = (a+1)b + c - 1$$

*for some positive integers $a$, $b$, and $c$ satisfying $a \leq c \leq k$ and $b + (c \bmod a) \leq k$.*

*Proof.* First, suppose $n = (a+1)b + c - 1$ for some positive integers $a$, $b$, and $c$ satisfying $a \leq c \leq k$ and $b + (c \bmod a) \leq k$. Let $r = c \bmod a$ and $q = (c-r)/a$ so that $c = aq + r$. Consider the partition $\kappa = [a+1, a+1, \ldots, a+1, a, a, a, \ldots, a, r]$ that contains $b$ parts equal to $a+1$, $q$ parts equal to $a$, and one part equal to $r$. Then $\kappa$ is a partition of $n+1$. Let $\lambda$ be the 1-minor of $\kappa$ formed by removing the corner square $X$ that appears in the last part in $\kappa$ equal to $a+1$, namely, $\kappa_b$, and let $\mu$ be the partition formed by removing the corner square $Y$ appearing in the last part equal to $a$, namely, $\kappa_{b+q}$. Then $|\operatorname{Out}_\kappa(X)| = aq + r = c \leq k$ and $|\operatorname{Out}_\kappa(Y)| = b + r = b + (c \bmod a) \leq k$. Hence, each of $|\operatorname{Out}_\mu(X)|$ and $|\operatorname{Out}_\lambda(Y)|$ is at most $k-1$, so by Corollary 3.4 we have $M_k(\lambda) = M_k(\mu)$.

Conversely, suppose $n$ and $k$ are such that there exist partitions $\lambda \neq \mu$ of $n$ with $M_k(\lambda) = M_k(\mu)$. We will show that we can find a partition $\kappa'$ of $n+1$ having two squares $X'$ and $Y'$ that are either in adjacent rows or adjacent columns in the Young diagram of $\kappa'$, and such that $|\operatorname{In}(X')| \geq n-k$ and $|\operatorname{In}(Y')| \geq n-k$.

By Lemma 3.5, there exists a partition $\nu$ of $n$ such that $d(\lambda,\nu) = 2$ and $M_k(\lambda) = M_k(\nu)$. Let $\kappa = \lambda \cup \nu$. Let $X$ be the square in $\kappa$ that lies outside of $\lambda$ and let $Y$ be the square in $\kappa$ that lies outside of $\nu$. Assume without loss of generality that $X$ lies above and to the right of $Y$. Suppose $X$ is in row $b$ and $Y$ is in column $a$, so that $\operatorname{In}(X) \cap \operatorname{In}(Y)$ is an $a \times b$ rectangle of squares. Let $c$ and $d$ be such that $Y$ is in row $b+c$ and $X$ is in column $a+d$. We may also assume without loss of generality that $a \leq b$, by interchanging the rows and columns if necessary.

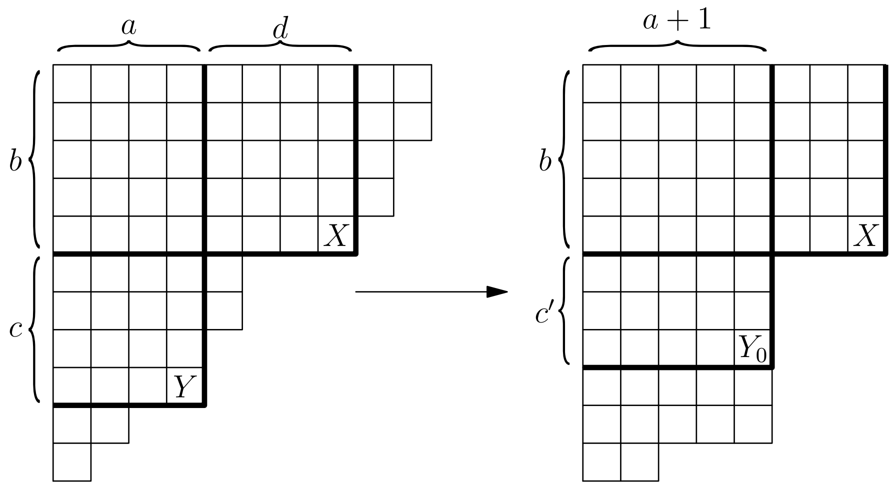



Figure 3: Moving $Y$ closer to $X$ as in the proof of Lemma 3.6.

If $c = 1$ or $d = 1$, then we can set $\kappa' = \kappa$. Otherwise, let $c'$ be the largest positive integer such that $(a+1)c' \leq ac$. By this definition, we have $(a+1)(c'+1) > ac$, so

$$(a+1)c' \geq ac - a. \tag{3.4}$$

Let $m = |\operatorname{Out}_\kappa(X)|$, and write $m = (a+1)q + r$ where $q$ and $r$ are nonnegative integers with $0 \leq r \leq a$. Define $\eta$ to be the partition having $b$ parts equal to $a+d$, followed by $q$ parts equal to $a+1$, and one part equal to $r$. In other words, we stack all of the squares in the outer region of $X$ in rows of $a+1$ (with $r$ left over) below row $b$, as in Figure 3. Let $Y_0$ be the last square in row $b+c'$ in $\eta$. Notice that $Y_0$ is closer to $X$ than $Y$ is, both vertically and horizontally.

We clearly have $|\operatorname{In}_\eta(X)| = |\operatorname{In}_\kappa(X)|$, which is at least $n-k$ by Lemma 3.3. Using

(3.4) and our assumption that $b \geq a$, we have

$$\begin{aligned} |\operatorname{In}_\eta(Y_0)| &= (a+1)(b+c') \\ &= (a+1)b+(a+1)c' \\ &\geq (a+1)b+ac-a \\ &= a(b+c)+b-a \\ &= |\operatorname{In}_\kappa(Y)|+b-a \\ &\geq |\operatorname{In}_\kappa(Y)|. \end{aligned}$$

Since $|\operatorname{In}_\kappa(Y)| \geq n-k$ by Lemma 3.3, we have $|\operatorname{In}_\eta(Y_0)| \geq n-k$.

We can now continue this process starting with $\eta$ and new values $a$ and $b$ formed by the intersection of the inner regions of $Y_0$ and $X$. Hence, we can form a partition $\kappa'$ having $X'$ and $Y'$ either in adjacent rows or adjacent columns and each of $|\operatorname{In}_{\kappa'}(X')|$ and $|\operatorname{In}_{\kappa'}(Y')|$ is at least $n-k$. This implies that each of $|\operatorname{Out}_{\kappa'}(X')|$ and $|\operatorname{Out}_{\kappa'}(Y')|$ is at most $k$.

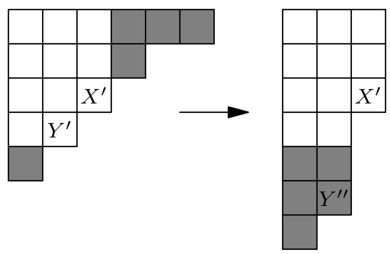



Figure 4: Creating $\kappa''$.

Finally, consider such a partition $\kappa'$. Suppose, without loss of generality (by interchanging rows and columns if necessary), that $X'$ and $Y'$ are in adjacent columns with $Y'$ below and to the left of $X'$, as in Figure 4. Let $p$ be the number of squares of the Young diagram that are in both of $\operatorname{Out}(X')$ and $\operatorname{Out}(Y')$. Suppose $Y'$ is in column $a$ of the Young diagram. Write $p = aq + r$ where $q$ and $r$ are nonnegative integers with $0 \leq r < a$. Consider the partition $\kappa''$ formed by removing all $p$ aforementioned squares from $\kappa'$, and then adding $q$ rows of $a$ and one row of $r$ squares below the row containing $Y'$. Let $Y''$ be the new corner square in column $a$. Then $\kappa''$ has the form $[a+1, a+1, \ldots, a+1, a, a, a, \ldots, a, r]$. Let $b$ be the number of parts of $\kappa''$ equal to $a+1$, and let $s$ be the number of parts equal to $a$. Define $c = as + r$. Then we see that $n + 1 = (a + 1)b + c$. Also, $a \leq c = as + r = p + a(s - q) = |\operatorname{Out}_{\kappa'}(X')| \leq k$, and $b + (c \bmod a) = b + r \leq b + p = |\operatorname{Out}_{\kappa'}(Y')| \leq k$.

Therefore, there exist partitions $\lambda \neq \mu$ of $n$ with $M_k(\lambda) = M_k(\mu)$ if and only if $n$ can be expressed in the form

$$n = (a+1)b + c - 1$$

for some positive integers $a$, $b$, and $c$ satisfying $a \leq c \leq k$ and $b + (c \bmod a) \leq k$. $\square$

Theorem 2.5 follows immediately, and we provide the proof below.

**Theorem 2.5.** Let $k$ be a positive integer. Then reconstructibility of partitions of $n$ from their $k$-minors holds for $n \geq k^2 + 2k$, and fails for $n = k^2 + 2k - 1$.

*Proof.* Let $n$ and $k$ be positive integers that satisfy the conditions in Lemma 3.6 for some $a$, $b$, and $c$. Then the inequality $b + (c \bmod a) \leq k$ implies $b \leq k$, so each of $a$, $b$, and $c$ are at most $k$. Hence, for a given $k$, the largest value of $n$ for which reconstructibility fails occurs when $a = b = c = k$ and $n = k^2 + 2k - 1$. □

We now introduce an auxiliary function, $h$, which we will later show is identical to $g$.

**Definition.** Let $n$ be a positive integer. Then $h(n)$ is the largest value $k$ for which partitions of $n$ can be reconstructed from their $k$-minors for all $k \leq h(n)$.

By this definition, partitions of $n$ cannot be reconstructed from their $(h(n) + 1)$-minors. We proceed to show that in fact there are no values $k$ larger than $h(n) + 1$ for which partitions of $n$ can be reconstructed from their $k$-minors.

**Lemma 3.7.** *Reconstructibility of partitions of $n$ from their $k$-minors holds if and only if $k \leq h(n)$.*

*Proof.* By the definition of $h$, the smallest positive integer $k$ for which partitions of $n$ cannot be reconstructed from their $k$-minors is $h(n) + 1$. Let $k_0 = h(n) + 1$. Then there exist partitions $\lambda$ and $\nu$ of $n$ such that $M_{k_0}(\lambda) = M_{k_0}(\nu)$. Let $k \geq k_0$ be arbitrary. Then $M_k(\lambda)$ consists of all $(k - k_0)$-minors of the elements of $M_{k_0}(\lambda)$, and similarly $M_k(\nu)$ consists of the $(k - k_0)$-minors of the elements of $M_{k_0}(\nu)$, so $M_k(\lambda) = M_k(\nu)$. Hence, reconstructibility fails for all $k \geq h(n) + 1$, and the claim follows. □

To prove Theorem 2.1, it now suffices to show that $h(n) = g(n)$ for all $n$. We first prove an intermediate lemma, which provides a formula for $h$.

**Lemma 3.8.** *Let $n$ be a positive integer, and let $S$ be the set of all solutions $(a, b, s, t)$ to the Diophantine equation $n = (a + 1)b + sa + t - 1$ for which $a$, $b$, $s$ are positive integers and $t$ is nonnegative. Then*

$$h(n) = \min_{(a,b,s,t) \in S} (\max\{sa + t, b + t\}) - 1.$$

*Proof.* Let $n$ be a positive integer. Recall that $h(n) + 1$ is the smallest value $k$ for which there are two partitions of $n$ having the same set of $k$-minors. By Lemma 3.6, this is the smallest value $k$ for which there exists a solution $(a, b, c)$ in positive integers to $n = (a + 1)b + c - 1$ satisfying $a \leq c \leq k$ and $b + (c \bmod a) \leq k$.

Let $R$ be the set of all solutions $(a, b, c)$ in positive integers to $n = (a + 1)b + c - 1$ for which $a \leq c$. Let $(a_1, b_1, c_1) \in R$. The smallest $k_1$ for which $c_1 \leq k_1$ and $b_1 + (c_1 \bmod a_1) \leq k_1$ is $k_1 = \max\{c_1, b_1 + (c_1 \bmod a_1)\}$. Hence, the smallest value $k$ for which two partitions of $n$ have the same set of $k$-minors is $\min_{(a,b,c) \in R}(\max\{c, b + (c \bmod a)\})$. It follows that $h(n) = \min_{(a,b,c) \in R}(\max\{c, b + (c \bmod a)\}) - 1$, and so it suffices to show that $\min_{(a,b,c) \in R}(\max\{c, b + (c \bmod a)\}) = \min_{(a,b,s,t) \in S}(\max\{sa + t, b + t\})$.

Let $m = \min_{(a,b,c)\in R}(\max\{c, b+(c \bmod a)\})$ and let $(a_0, b_0, c_0) \in R$ such that $\max\{c_0, b_0+(c_0 \bmod a)\} = m$. Since $a_0 \le c_0$ by the definition of $R$, we can write $c_0 = s_0a_0 + t_0$ for some positive integer $s_0$ and nonnegative integer $t_0 < a_0$. Then $(a_0, b_0, s_0, t_0) \in S$. Furthermore, $t_0 = c_0 \bmod a_0$, so $\max\{s_0a_0 + t_0, b_0 + t_0\} = \max\{c_0, b_0 + (c_0 \bmod a)\} = m$. Hence $m$ is attained as a value of $\max\{sa + t, b + t\}$ for some $(a, b, s, t) \in S$.

Finally, assume that there exists a solution $(a, b, s, t) \in S$ such that $\max\{sa+t, b+t\} < m$. Write $sa + t = s'a + t'$ where $s'$ is a positive integer and $0 \le t' < a$. Then $t' \le t$, so $b + t' \le b + t$. It follows that $\max\{s'a + t', b + t'\} \le \max\{sa + t, b + t\} < m$. Let $c = s'a+t'$. Then $t' = c \bmod a$ and $a \le c$, so $(a, b, c) \in R$ satisfies $\max\{c, b+(c \bmod a)\} = \max\{s'a + t', b + t'\} < m$. This is a contradiction since $m$ is the minimum possible value of $\max\{s'a + t', b + t'\}$. Hence $m = \min_{(a,b,s,t)\in S}(\max\{sa + t, b + t\})$ as desired. □

We finally have the tools to prove our main result.

**Theorem 2.1.** Let $n$ and $k$ be positive integers with $k < n$. For any positive integer $n \notin \{5, 12, 21, 32\}$, define

$$g(n) = \min_{0\le t\le n} \rho(n + 2 - t) - 2 + t$$

and also define $g(5) = 1$, $g(12) = 3$, $g(21) = 5$, $g(32) = 7$. Then partitions of $n$ can be reconstructed from their sets of $k$-minors if and only if $k \le g(n)$.

*Proof.* We wish to show that $g(n) = h(n)$ for all $n \ge 2$. By a straightforward calculation, this holds for $n = 2$, 3, 5, 9, 12, 21, and 32.

For a fixed positive integer $N$, consider the Diophantine equation

$$N = (a + 1)b + sa + t - 1 \tag{3.5}$$

in positive integers $a, b, s$ and nonnegative integers $t$. Define a *minimal solution* to this equation to be a solution $(a, b, s, t)$ for which the value $\max\{sa + t, b + t\} - 1$ attains its minimum. Notice that if $N = n$ and $s = 1$, we have $n - t + 2 = (a + 1)(b + 1)$. Hence, for any nonnegative integer $t$ we have $\min(\max\{sa + t, b + t\}) - 1 = \min(\max\{a + 1, b + 1\}) - 2 + t = \rho(n + 2 - t) - 2 + t$, where the minimum is taken over all $a$ and $b$ satisfying $n - t + 2 = (a + 1)(b + 1)$. Taking the smallest such value over all nonnegative integers $t$, we see that if there exists a minimal solution to (3.5) having $s = 1$ then the minimum value is $g(n)$, and so $h(n) = g(n)$ by Lemma 3.8. Hence, it suffices to show that a minimal solution to (3.5) having $s$ as small as possible has $s = 1$.

To do so, we use induction on $n$. As base cases, it is easily verified that this property holds for $n = 4$, 6, 10, 13, 22, and 33.

Now, let $n \notin \{1, 2, 3, 4, 5, 6, 9, 10, 12, 13, 21, 22, 32, 33\}$ be an arbitrary positive integer and assume that one of the minimal solutions to (3.5) for $N = n - 1$ having $s$ as small as possible has $s = 1$, and hence that $h(n - 1) = g(n - 1)$. Let $(a, b, s, t)$ be a minimal solution to (3.5) for $N = n$ having $s$ as small as possible. We show that $s = 1$.

First, suppose $t \ge 1$. Then $(a, b, s, t - 1)$ is a solution to (3.5) when $N = n - 1$. We claim that this must be a minimal solution for $N = n-1$. For, assume that $(a', b', s', t')$ is a solution for $N = n-1$ satisfying $\max\{s'a'+t', b'+t'\} < \max\{sa+t-1, b+t-1\}$. Then $(a', b', s', t'+1)$ is a solution for $N = n$ and $\max\{s'a'+t'+1, b'+t'+1\} < \max\{sa+t, b+t\}$, which is impossible since we assumed that $(a, b, s, t)$ was minimal for $N = n$. Thus, since $(a, b, s, t-1)$ is a minimal solution for $N = n-1$, we have $s = 1$ by the inductive hypothesis.

Now, suppose $t = 0$, so that $n = ab + b + sa - 1$, and assume to the contrary that $s \geq 2$. We consider several cases.

**Case 1.** Suppose $sa + 2 \leq b$. Then $a \leq (b-2)/s$ and $n = (a+1)b + sa - 1$, so we have

$$\begin{aligned} n &\leq ((b-2)/s + 1)b + s(b-2)/s - 1 \\ sn &\leq (b - 2 + s)b + sb - 2s - s \\ (s-1)^2 + sn + 3s &\leq b^2 + 2(s-1)b + (s-1)^2 \\ \sqrt{sn + s^2 + s + 1} &\leq b + (s-1) \\ 1 - s + \sqrt{sn + s^2 + s + 1} &\leq b \end{aligned}$$

It is straightforward to verify that for a fixed $n$, the expression $1 - s + \sqrt{sn + s^2 + s + 1}$ is an increasing function of $s$ for $s > 0$, so using our assumption that $s \geq 2$ we have $1 - 2 + \sqrt{2n + 2^2 + 2 + 1} \leq b$. Hence

$$b \geq \sqrt{2n+7} - 1.$$

Since $sa + 2 \leq b$, we have $h(n) = \max\{sa, b\} = b$. Notice that if $(a, b, s, t)$ is a solution to (3.5) for $N = n - 1$, then $(a, b, s, t+1)$ is a solution to (3.5) for $N = n$, and so by Lemma 3.8 we have $h(n) \leq h(n-1) + 1$. Thus, to obtain a contradiction it suffices to show that $b \geq h(n-1) + 2$. Notice that the function $\sqrt{2n+7} - 1$ has a larger order of growth than $\sqrt{n+1} + 3\sqrt[4]{n+1} + 2$. The inequality $\sqrt{2n+7} - 1 \geq \sqrt{n+1} + 3\sqrt[4]{n+1} + 2$ holds for all $n \geq 4360$. It follows that for $n \geq 4360$, we have $b \geq \sqrt{n+1} + 3\sqrt[4]{n+1} + 2 \geq g(n-1) + 2 = h(n-1) + 2$ by Theorem 2.4 and the inductive hypothesis. Hence $b > h(n)$, a contradiction.

For $n < 4360$, a computer calculation shows that in fact $g(n-1) \leq \sqrt{n+1} + 3\sqrt[4]{n+1} - 5$. In addition, the inequality $\sqrt{2n+7} - 1 \geq \sqrt{n+1} + 3\sqrt[4]{n+1} - 3$ holds for all $n \geq 1765$. By a similar argument to that above, we now have that if $n \geq 1765$ then $b > h(n)$.

For $n < 1765$, a computer calculation verifies that no minimal solution has $sa + 2 \leq b$ and $s \geq 2$.

In each of the remaining cases, we will show that there is a solution $(a', b', s', t')$ to (3.5) having $\max\{s'a' + t', b' + t'\} \leq \max\{sa, b\}$ and $1 \leq s' < s$, so that we can decrease $s$ and still form a minimal solution, thereby obtaining a contradiction.

**Case 2.** Suppose $a + 2 \leq b \leq sa + 1$. Then we can rewrite $n$ as follows.

$$\begin{aligned} n &= ab + b + sa - 1 \\ &= (a+1)(b-1) + sa + a \\ &= (a+1)(b-1) + (b-1) + (sa + a - b + 2) - 1 \\ &= a'b' + b' + s'a' + t' - 1 \end{aligned}$$

where $a' = a+1$, $b' = b-1$, and $s'a' + t' = sa + a - b + 2$ such that $0 \leq t' < a'$.

To show that $1 \leq s'$, note that since $b \leq sa+1$ we have $a+1 \leq sa+a-b+2 = s'(a+1)+t'$. Furthermore, since $a+2 \leq b$, we have $s'a' \leq s'a'+t' = sa+(a+2)-b \leq sa < sa'$, so $s' < s$.

Finally, we have $s'a' + t' \leq sa$ and also $b' + t' = (b-1) + (s'a' + t') - s'a' = (b-1) + (sa + a - b + 2) - s'(a+1) \leq (b-1) + (sa + a - b + 2) - (a+1) = sa$, so $\max\{s'a' + t', b' + t'\} \leq sa \leq \max\{sa, b\}$ as desired.

**Case 3.** Suppose $b \leq a+1$ and $s \geq 3$. We write

$$\begin{aligned} n &= ab + b + sa - 1 \\ &= a(b+1) + (b+1) + (sa - a - 1) - 1 \\ &= a'b' + b' + s'a' + t' - 1 \end{aligned}$$

where $a' = a$, $b' = b+1$, and $s'a' + t' = sa - a - 1$ such that $0 \leq t' < a'$.

To show $1 \leq s'$, note that since $s \geq 3$ and $a \geq 1$ we have $a \leq 3a-(a+1) \leq sa-a-1 = s'a + t'$. Furthermore, since $s'a' + t' < sa'$, we must have $s' < s$.

Finally, we have $s'a' + t' \leq sa' = sa$ and also $b' + t' \leq b+1+a-1 = b+a \leq 2a+1 \leq 3a \leq sa$, so $\max\{s'a' + t', b' + t'\} \leq sa \leq \max\{sa, b\}$ as desired.

**Case 4.** Suppose $b \leq a-1$ and $s = 2$. We write

$$\begin{aligned} n &= ab + b + sa - 1 \\ &= (a+1)b + b + (2a - b) - 1 \\ &= a'b' + b' + s'a' + t' - 1 \end{aligned}$$

where $a' = a+1$, $b' = b$, and $s'a' + t' = 2a - b$ such that $0 \leq t' < a'$.

To show $1 \leq s'$, note that since $b \leq a-1$ we have $a+1 \leq 2a-b = s'a'+t' = s'(a+1)+t'$. Furthermore, since $s'(a+1) + t' = 2a - b < 2(a+1)$ we have $s' \leq 1$. Hence $s' = 1$ (and therefore $1 \leq s' < s$).

Finally, we have $s'a' + t' < 2a = sa$ and also $b' + t' \leq a - 1 + a < 2a = sa$, so $\max\{s'a' + t', b' + t'\} \leq sa \leq \max\{sa, b\}$ as desired.

In the remaining two cases, we finally discover the significance of the seemingly mysterious values 3, 5, 9, 12, 21, and 32.

**Case 5.** Suppose $b = a$ and $s = 2$. Then

$$\begin{aligned} n &= ab + b + sa - 1 \\ &= a^2 + 3a - 1 \\ &= (a+2)(a-1) + 2a + 1 \\ &= (a+2)(a-1) + (a-1) + ((a+2)+1) - 1 \\ &= a'b' + b' + s'a' + t' - 1 \end{aligned}$$

where $a' = a+2$, $b' = a-1$, and $s' = t' = 1$. Clearly $1 \leq s' < s$.

Notice that $b' + t' = a < 2a = sa$, and $s'a' + t' = a + 3 \leq 2a$ if and only if $a \geq 3$, so when $a \geq 3$ we have $\max\{s'a' + t', b' + t'\} \leq sa \leq \max\{sa, b\}$. If $a = 1$ or $a = 2$, we obtain the extraneous values $n = 3$ and $n = 9$.

**Case 6.** Suppose $b = a + 1$ and $s = 2$. Then

$$\begin{aligned} n &= ab + b + sa - 1 \\ &= a^2 + 4a \\ &= (a-1)(a+3) + 2a + 3 \\ &= (a-1)(a+3) + (a+3) + (a+1) - 1 \\ &= a'b' + b' + s'a' + t' - 1 \end{aligned}$$

where $a' = a-1$, $b' = a+3$, $s' = 1$, and $t' = 2$. For $a \geq 5$, we have $s'a'+t' = a+1 \leq 2a = sa$ and $b' + t' \leq a + 3 + 2 \leq 2a = sa$, and so $\max\{s'a' + t', b' + t'\} \leq sa \leq \max\{sa, b\}$. For $a = 1$, 2, 3, and 4, we obtain the extraneous values $n = 5$, 12, 21, and 32, for which every minimal solution to (3.5) has $s \geq 2$.

This completes the proof. □

# 4 Applications and Future Work

In this section, we present two direct applications of our results and propose a natural extension of the partition reconstruction problem to Young tableaux.

## 4.1 The Character Reconstruction Problem for $S_n$

Suppose $G$ is a finite group and $\mathcal{H}$ is a collection of subgroups of $G$. For any representation $x$ of $G$ over a field of characteristic zero, define $\mathrm{Irr}\,(x)$ to be the set of irreducible representations appearing as composition factors in the decomposition of $x$ into irreducible representations. Similarly, if $\chi$ is the character corresponding to $x$, define $\mathrm{Irr}\,(\chi)$ to be the set of irreducible characters corresponding to the elements of $\mathrm{Irr}(x)$. The equivalence relation $\sim_{\mathcal{H}}$ on the irreducible representations of $G$ is defined by $x \sim_{\mathcal{H}} y$ if and only if $\mathrm{Irr}\,(x|_H) = \mathrm{Irr}\,(y|_H)$ for all $H \in \mathcal{H}$, where $x|_H$ denotes the restriction of $x$ to $H$. The equivalence $\chi \sim_{\mathcal{H}} \phi$ is defined in a similar manner for irreducible characters $\chi$ and $\phi$ of $G$.

The character reconstruction problem for finite groups is stated in [8] as follows. For which collections $\mathcal{H}$ does $\chi \sim_{\mathcal{H}} \phi$ imply that $\chi = \phi$ for any two irreducible characters $\chi$ and $\phi$ of $G$?

Consider the symmetric group $S_n$, the group of permutations of $\{1, 2, \ldots, n\}$. It is well known that there is a one-to-one correspondence between irreducible representations of $S_n$ and partitions of $n$. (See [4], [5], or [11] for a more detailed discussion of the representation theory of the symmetric groups.) There is a natural way to construct this correspondence such that if $H$ is the stabilizer of some $k$-element subset of $\{1, 2, \ldots, n\}$, then the representation $x^\lambda$ associated with a partition $\lambda$ satisfies $\mathrm{Irr}(x^\lambda|_H) = \{x^\mu : \mu \in M_k(\lambda)\}$. This is known as the Branching Theorem.

Now, suppose $\mathcal{H}$ consists of a single subgroup, $H$, which stabilizes some $k$ points in $\{1, 2, \ldots, n\}$. It follows from the Branching Theorem that an irreducible representation $\chi^\lambda$ of $S_n$ can be recovered from its restriction to $H$ if and only if $\lambda$ can be reconstructed from its set of $k$-minors. This argument holds for characters as well as representations, so we obtain the following corollary to Theorem 2.1.

**Corollary 4.1.** *Let $n$ and $k$ be positive integers with $k < n$, and let $H \subset S_n$ be the stabilizer of a $k$-element subset of $\{1, 2, \ldots, n\}$. Then any irreducible representation $x$ of $S_n$ (and hence its character $\chi$) can be reconstructed from the set of irreducible composition factors of $x|_H$ if and only if $k \leq g(n)$.*

## 4.2 An Application to Permutation Reconstruction

A *$k$-reduction* of a permutation $p = p_1p_2 \ldots p_n$ of $\{1, 2, \ldots, n\}$ is a permutation of $\{1, 2, \ldots, n-k\}$ formed by re-numbering the elements of an $(n-k)$-element subsequence of $p_1p_2 \ldots p_n$ such that the relative order of the elements is preserved. For instance, 132 is a 2-reduction of the permutation $p = 31524$ by considering the subsequence 154 of $p$. The problem of reconstructing permutations from certain sets or multisets of $k$-reductions has been of much recent interest ([1], [2], [3], [9], [10]).

A natural variant on this problem that is less well understood is the reconstruction of permutations from their *cycle $k$-minors*. Given a permutation written as a product of disjoint cycles, a cycle $k$-minor is formed by removing some $k$ of the elements and re-numbering the remaining elements so as to preserve their order. For example, (315)(24) is a cycle 1-minor of (4162)(35), formed by deleting the 2 from the cycle (4162), and then subtracting 1 from every remaining number that is larger than 2. It has been shown [7] that all permutations in $S_n$ can be reconstructed from their sets of cycle 1-minors if and only if $n \geq 6$, and conjectures that for any positive integer $k$, we can reconstruct permutations in $S_n$ from their cycle $k$-minors for sufficiently large $n$.

Theorem 2.1 provides an interesting insight into this problem. Recall that the conjugacy classes of $S_n$ consist of all permutations which are a product of disjoint cycles having a given number of cycles of each length. Hence we can associate each partition $\lambda$ of $n$ with the conjugacy class in $S_n$ consisting of the permutations $p$ having one $\lambda_i$-cycle for each $i$ in the decomposition of $p$ into disjoint cycles. For example, the partition $[3, 3, 2]$ is associated with the permutations in $S_8$ having disjoint cycle decomposition of the form $(abc)(def)(gh)$.

Clearly, the partition associated with a cycle 1-minor of a permutation $p$ is a 1-minor of the partition associated with $p$. Thus we have the following corollary to Theorem 2.1.

**Corollary 4.2.** *The conjugacy class of a permutation can be reconstructed from its set of cycle $k$-minors whenever $k \leq g(n)$.*

In the case $k = 1$, this is not sufficient to reconstruct the permutation as well, since reconstructibility holds for $n \geq 3$ for partitions, whereas $n \geq 6$ is required to reconstruct permutations from their cycle 1-minors. Nevertheless, this may be a useful intermediate step in solving this conjecture.

### 4.3 Reconstructing Young Tableaux

Having solved the partition reconstruction problem, it would be interesting to extend this question to Young tableaux, which also arise naturally in representation theory. A (standard) *Young tableau* of size $n$ is a Young diagram of a partition of $n$ whose squares are labeled with the numbers $1, 2, \dots, n$ such that the labels are increasing from left to right in each row and from top to bottom in each column.

We propose a natural definition of a minor of a Young tableau inspired by *jeu de taquin*, or "the teasing game." (See [11], pp. 419-425.) Suppose we remove a square $X$ and its label from a Young tableau and re-number the squares from 1 to $n-1$, again preserving the relative order of the labels. If $X$ was a corner square, we are left with a tableau of size $n-1$. Otherwise, consider the square $Y$ directly to the right of $X$ and the square $Z$ below $X$ (note that either of $Y$ or $Z$ may not exist). If $Y$ has a smaller label than $Z$ or $Z$ does not exist, slide $Y$ to the left, and otherwise slide $Z$ up to fill in the missing square. Continue this sliding process until a new tableau is formed. We define this to be a 1-minor of the tableau, and similarly define a $k$-minor to be a tableau formed by taking $k$ successive 1-minors.

Theorem 2.1 shows that we can reconstruct the *shape* of the tableau from its set of $k$-minors whenever $k \leq g(n)$, since every possible $k$-minor of the corresponding partition will appear as the shape of some $k$-minor formed by removing corner squares in succession. However, this is not always sufficient to reconstruct the labeling of the squares. For example, the two tableaux of size 4 shown below have the same set of 1-minors. This prompts the question of which $n$ and $k$ have the property that any tableau with $n$ squares can be reconstructed from its set of $k$-minors.

| | |
|---|---|
| 1 | 3 |
| 2 | 4 |

| | |
|---|---|
| 1 | 2 |
| 3 | 4 |

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